Backbone colouring: tree backbones with small diameter in planar graphs

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February 22, 2013

Abstract

Given a graph G and a spanning subgraph T of G, a backbone k-colouring for (G,T) is a mapping $c : V(G) \to \{1,\dots,k\}$ such that $|c(u) - c(v)| \geq 2$ for every edge $uv \in E(T)$ and $|c(u) - c(v)| \geq 1$ for every edge $uv \in E(G) \setminus E(T)$. The backbone chromatic number $BBC(G,T)$ is the smallest integer k such that there exists a backbone k-colouring of (G,T). In 2007, Broersma et al. [2] conjectured that $BBC(G,T) \leq 6$ for every planar graph G and every spanning tree T of G. In this paper, we prove this conjecture when T has diameter at most four.

Keywords: Backbone colouring, planar graphs, Broerma's conjecture.

1 Introduction

All the graphs considered in this paper are simple. Let $G = (V,E)$ be a graph, and let $H = (V, E(H))$ be a spanning subgraph of G. A *k-colouring* of G is a mapping $f : V \to \{1,2,\dots,k\}$. Let f be a k-colouring of G. It is a *proper colouring* if $|f(u) - f(v)| \geq 1$. It is a *backbone colouring* for (G,H) if f is a proper colouring of G and $|f(u) - f(v)| \geq 2$ for all edges $uv \in E(H)$. The *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a proper k-colouring of G. The *backbone colouring number* $BBC(G,H)$ is the smallest integer k for which there exists a backbone k-colouring of (G,H).

If f is a proper k-colouring of G, then g defined by $g(v) = 2f(v) - 1$ is a backbone $(2k-1)$-colouring of (G,H) for any spanning subgraph H of G. Hence, $BBC(G,H) \leq 2\chi(G) - 1$. In [1, 2], Broersma et al. showed that for any integer k there is a graph G with a spanning tree T such that $BBC(G,T) = 2k - 1$.

The above inequality and the Four Colour Theorem implies that for any planar graph G and spanning subgraph H then $BBC(G,H) \leq 7$. However Broersma et al. [2] conjectured that this is not best possible if T is a tree.

Conjecture 1 *If G is a planar graph and T a spanning tree of G, then $BBC(G,T) \leq 6$.*

If true this conjecture would be best possible. Broersma et al. [2] gave an example of a graph G^* with a spanning tree T^* such that $BBC(G,T) = 6$. See Figure 1.

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†Projet Mascotte, I3S (CNRS, UNS) and INRIA, Sophia Antipolis. Partly supported by the French *Agence Nationale de la Recherche* under Grant GRATEL ANR-09-blan-0373-01. Email: Frederic.Havet@inria.fr

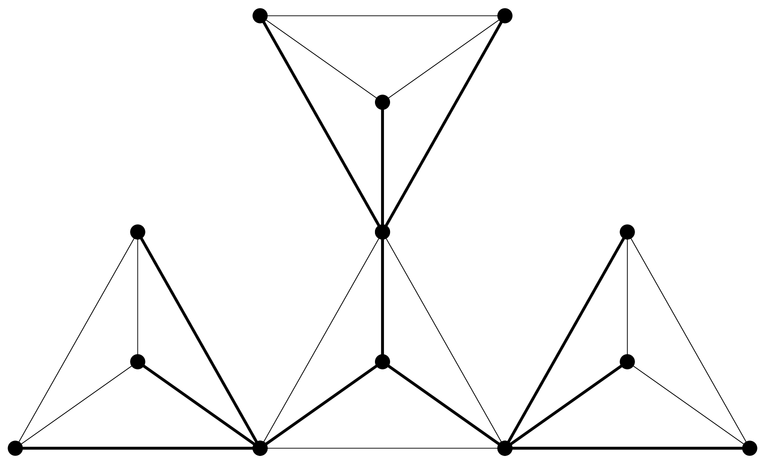

Figure 1: A planar graph G^* with a spanning tree T^* (bold edges) such that $BBC(G^*, T^*) = 6$.

Bu and Zhang [5] proved that, if G is a connected non-bipartite C_4-free planar graph, then there exists a spanning tree T of G such that $BBC(G, T) = 4$. On the other hand, Bu and Li [4] proved that, if G is a connected planar graph that is C_6-free or C_7-free and without adjacent triangles, then there exists a spanning tree T of G such that $BBC(G, T) \leq 4$. In [7], Wang et al. investigated backbone colouring for special graph classes such as Halin graphs, complete graphs, wheels, graphs with small maximum average degree and graphs with maximum degree 3.

The *diameter* of a graph is the maximum distance between two vertices in this graph. If T has diameter 2, then it is a *star*, that is a tree in which a vertex v, called the *center*, is adjacent to every other. If a planar graph G has a spanning star T, with center v, then $G - v$ is an outerplanar graph which can be properly 3-coloured with $\{1, 2, 3\}$. Thus assigning the colour 5 to v, we obtain a backbone 5-colouring of (G, T). This result may be extended if G has a spanning tree with diameter at most 3.

Proposition 2 *Let G be a planar graph with a spanning tree T. If T has diameter at most three, then $BBC(G, T) \leq 5$.*

Proof. Free to add some edges, we may assume that G is triangulated. If T has diameter at most 3, then there exists two adjacent vertices x and y such that all edges of T are incident to x or y. Let $z_1, \ldots, z_p$ be the common neighbours of x and y, ordered in clockwise order around x (and so in anti-clockwise order around y). We consider an embedding of G with outer face xyz_1.

For $1 \leq i \leq p - 1$, let G_i be the graph induced by the vertices in the cycle $xz_iyz_{i+1}x$ and inside, and let $H_i = G_i \setminus \{x, y\}$. Since G is triangulated, all the vertices are in at least one G_i. Furthermore, every H_i is outerplanar, and every vertex in $V(H_i) \setminus \{z_i, z_{i+1}\}$ is adjacent to exactly one of x, y.

We shall now define a backbone 5-colouring c of (G, T).

First, we set $c(x) = 1$, $c(y) = 5$ and $c(z_1) = 3$. Next, we extend this colouring to the H_i one after another. Since H_i is outerplanar, it is 3-colourable. Let c_i be a proper 3-colouring of H_i in $\{2, 3, 4\}$ such that $c_i(z_i) = c(z_i)$ and $c_i(z_{i+1}) \in \{3, 4\}$ if $z_{i+1}x \in E(T)$ and $c_i(z_{i+1}) \in \{2, 3\}$

if $z_{i+1}y \in E(T)$. We set $c(z_{i+1}) = c_i(z_{i+1})$, and for every vertex v of $V(H_i) \setminus \{z_i, z_{i+1}\}$, we define

- $c(v) = c_i(v)$, if $c_i(v) = 3$, or $c_i(v) = 2$ and $vy \in E(T)$, or $c_i(v) = 4$ and $vx \in E(T)$;
- $c(v) = 5$, if $c_i(v) = 2$ and $vx \in E(T)$;
- $c(v) = 1$, if $c_i(v) = 4$ and $vy \in E(T)$.

It is easy to check that c is a backbone 5-colouring of (G, T). □

Remark 3 Notice that the proof of Proposition 2 contains a explicit polynomial time algorithm to obtain a backbone 5-colouring of (G, T) when G is planar and T has diameter at most 3, since 3-colourings of outerplanar graphs can be obtained in polynomial time [6]. Proposition 2 is best possible, because when G is a complete graph on four vertices and T a spanning star of G, $BBC(G, T) = 5$.

In this paper, we settle Conjecture 1 for tree with diameter at most 4.

Theorem 4 *Let G be a planar graph with a spanning tree T. If T has diameter at most 4, then $BBC(G, T) \leq 6$.*

Note that this result is best possible as the tree T^* in the above example has diameter 4.

In the next section, we outline the proof of Theorem 4 whose details are postponed to Section 3.

2 The proof

We denote by Z_6 the set $\{1, 2, 3, 4, 5, 6\}$ and, for any integer $a \in Z_6$, we denote by $[a]$ the set $\{a-1, a, a+1\} \cap Z_6$.

Let $G = (V, E)$ be a planar graph and T a spanning tree of G with diameter at most 4. T has a vertex r such that every vertex is at distance two from it in T. We call such a vertex the *root* of T. A vertex of $V \setminus \{r\}$, is a *twig* if it is adjacent to r in T and a *leaf* otherwise.

We shall prove a slightly stronger result than the one of Theorem 4.

Theorem 5 (G, T) *admits a backbone colouring in Z_6 such that the root is assigned* 1.

Proof. In the remaining, by (G, T)-*colouring*, one should understand a backbone 6-colouring of (G, T) such that r is assigned 1.

We will prove it by considering a minimum counterexample (G, T) with respect to its number of vertices. An edge of $E \setminus E(T)$ is said to be *thin*. Free to add some more thin edges, we may assume that G is triangulated.

If T has a unique twig, then it has diameter 2, and we have the result by the proof of Proposition 2. (The root corresponds to x_1 and the twig to x_2.) Hence T has at least two twigs. We consider an embedding of G in the plane such that the outer face contains r and a minimum number of thin edges.

The *interior* (resp. *exterior*) of a cycle C, denoted C^{int} (resp. C^{ext}) is the subgraph of G induced by C and the vertices inside C (resp. outside C).

Let e be a thin edge. The graph $T \cup \{e\}$ has a unique cycle C_e (which contains e). The edge e is *overstepping* if there is a vertex inside C_e. In other words, $V(C_e^{int}) \neq V(C_e)$. Let O be the set of overstepping edges. There is a partial order $\leq$ on O defined as follows: $e_1 \leq e_2$ if $e_1 = e_2$ or e_1 is inside C_{e_2} (Lemma 6 proves that $\leq$ is a partial order). Observe that the Hasse diagram of such a partial order is a set of at most two disjoint trees, each one rooted at an overstepping thin edge in the outer face. Indeed, it is easy to see that every overstepping edge e that is not maximal has a unique *successor* for $\leq$ (i.e. overstepping edge f such that if $e \leq e' \leq f$ then $e' \in \{e, f\}$). This successor is one of the two edges of the face containing e contained in C_e^{ext}. Furthermore, every edge e has at most two *predecessors* for $\leq$: the two other edges of the face containing e contained in C_e^{int}.

The idea of the proof is to find a "good" overstepping edge e, such that a backbone 6-colouring of the graph induced by $V(C_e^{ext})$ (which exists by minimality of (G, T)) can be extended to $V(C_e^{int})$ to obtain a (G, T)-colouring. This will be a contradiction.

Natural candidates for such a good edge are overstepping edges e which are *minimal* for $\leq$ (i.e. such that $e' \leq e$ implies $e' = e$) or their sucessors. However we will need to consider a more precise partial ordering. If there are two overstepping edges $e_3 = rv_1$ and $e_4 = v_1v_2$ such that v_1 and v_2 are leaves and $e_4 \not\leq e_3$, (i.e. e_4 is not inside e_3), then we would like to have e_3 smaller than e_4 in the ordering.

This leads to the following binary relation $\preceq$ between overstepping edges: $e_1 \preceq e_2$ if $e_1 \leq e_2$ or there exist two edges $e_3 = rv_1$ and $e_4 = v_1v_2$ such that v_1 and v_2 are leaves, $e_4 \not\leq e_3$, $e_1 \leq e_3$ and $e_4 \leq e_2$. In Lemma 6, we prove that $\preceq$ is a partial order.

In the remainder of the paper, we will only consider the partial order $\preceq$. Hence the terms minimal, predecessor, successor, and so on refer to $\preceq$.

We first show some properties of minimal overstepping edges and deduce in Lemma 14 that if e is a minimal overstepping edge, then C_e^{int} is isomorphic to one of the graphs A_1, A_2 or A_3, depicted in Figure 2. In addition, if $C_e^{int} = A_1$, then $rv_1 \in E(G)$.

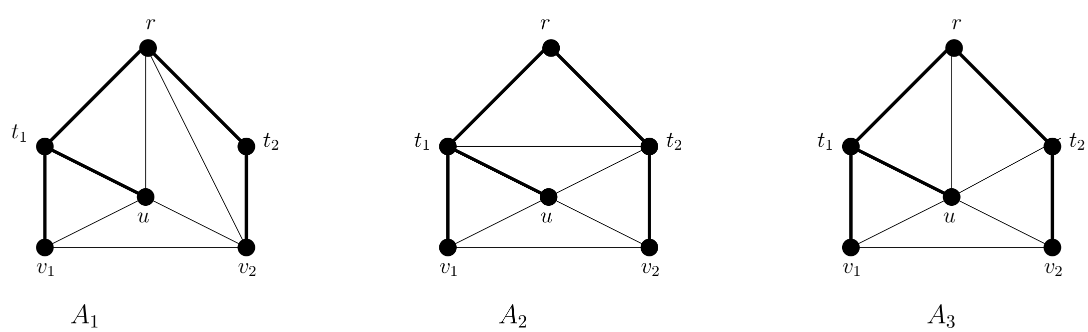

Figure 2: Configurations A_1, A_2 and A_3

As any ordering, $\preceq$ may be decomposed into levels. The first level L_1 the *maximal* edges for $\preceq$ (i.e. such that $e \preceq e'$ implies $e' = e$). This level contains at most two edges, depending on the number of thin overstepping edges in the outer face. Then, for every $j \geq 1$, the level L_{j+1} is the set of predecessors of elements of L_j. The *depth* of $\preceq$, denoted D, is the maximum j such that L_j is not empty. An overstepping edge of L_D is said to be *ultimate*. An edge of L_{D-1} having at least one (ultimate) predecessor is said to be *penultimate*. An edge of L_{D-2} having at least one penultimate predecessor is said to be *antepenultimate*.

If f is a penultimate edge, then it has one or two predecessors. Furthermore each of this predecessors e is ultimate and so minimal. Thus C_e^{int} is isomorphic to A_1, A_2 or A_3. Analyzing all possible cases, we show (Corollary 17) that, if f is a penultimate edge, then C_f^{int} is isomorphic to B_1 or B_2, and that moreover $rv_1 \in E(G)$ and, if $C_f^{int} = B_2$, $rv_3 \notin E(G)$.

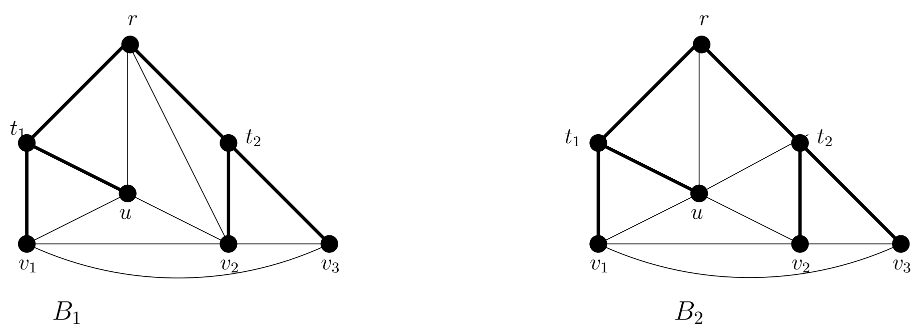

Figure 3: Configurations B_1 and B_2

Now if g is an antepenultimate edge, then it has one or two predecessors. Furthermore at least one of its predecessors f is penultimate (and so C_f^{int} is isomorphic to B_1 or B_2), and the other predecessor f' (if it exists) is either penultimate (so $C_{f'}^{int}$ is isomorphic to B_1 or B_2) or ultimate (so $C_{f'}^{int}$ is isomorphic to A_1, A_2 or A_3). Analyzing all the possibles cases again, we show that there are no antepenultimate edges (Corollary 24).

Now, suppose that G contains at least one overstepping edge. If e is a minimal edge, then C_e^{int} is isomorphic to some configuration A_i. In any of these cases, there is at least one face containing the root and only one thin edge. Therefore, the partial order considered contains a unique maximal overstepping edge e_0. Furthermore, since e_0 is not antepenultimate, $C_{e_0}^{int}$ must be isomorphic to one of the A_i or B_j configurations. We get a contradiction as the outer face contains r and the endpoints of e_0 and e_0 is the unique thin edge in this configuration and T would not be a tree.

We proved that G contains no overstepping edge. If the outer face of G contains only one thin edge, then G contains three vertices and the diameter of G is 2. If the outer face contains two thin edges e_1 and e_2, then one thin edge (say e_1) is adjacent to r, since r is on the outer face, and the other (say e_2) is adjacent to a twig t while both are incident to a vertex v in the outer face. Now, both r and v have a twig v' as a common neighbour through edges of T as T is a spanning tree. Since neither e_1 nor e_2 are overstepping, then $V(G) = \{r, t, v, v'\}$ and G has diameter 3. Both of these cases are solved using Proposition 2 and both can give colour 1 to the root, a contradiction. □

3 The details

Lemma 6 *The binary relation $\preceq$ is a partial order.*

Proof. Let e_0, e_1, e_2 be overstepping edges. At first, we prove that $\leq$ is a partial order. By the definition, it is clearly reflexive. Now suppose that $e_1 \leq e_2$ and $e_2 \leq e_1$. Then, by the definition, e_1 is inside C_{e_2} and e_2 is inside C_{e_1}. Clearly, this is only possible if $e_1 = e_2$. Then, $\leq$ is antisymmetric. Now suppose that $e_0 \leq e_1 \leq e_2$. Then e_0 is inside C_{e_1} and e_1 is inside C_{e_2}. This implies that C_{e_1} is inside C_{e_2}, and consequenlty e_0 is inside C_{e_2}. Then $e_0 \leq e_2$ and $\leq$ is transitive.

Now we prove that $\preceq$ is a partial order. Since $e_1 \leq e_2$ implies that $e_1 \preceq e_2$, then $\preceq$ is reflexive.

We claim that $\preceq$ is antisymmetric. To prove this, suppose that $e_1 \preceq e_2$ and $e_2 \preceq e_1$. If $e_1 \leq e_2$ and $e_2 \leq e_1$, then $e_1 = e_2$, since $\leq$ is antisymmetric. So, assume that $e_1 \not\leq e_2$ and $e_2 \leq e_1$. Since $e_1 \preceq e_2$, we have by the definition of $\preceq$ that there exist two overstepping edges $e_3 = rv_1$ and $e_4 = v_1v_2$ such that v_1 and v_2 are leaves, $e_4 \not\leq e_3$, $e_1 \leq e_3$ and $e_4 \leq e_2$. Then $e_4 \leq e_2 \leq e_1 \leq e_3$ and, by transitivity, $e_4 \leq e_3$, a contradiction.

Now assume that $e_1 \not\leq e_2$ and $e_2 \not\leq e_1$. Since $e_1 \preceq e_2$ and $e_2 \preceq e_1$, we have by the definition of $\preceq$ that there exist four overstepping edges $e_3 = rv_1$, $e_4 = v_1v_2$, $e_5 = rw_1$ and $e_6 = w_1w_2$ such that v_1, v_2, w_1, w_2 are leaves, $e_4 \not\leq e_3$, $e_1 \leq e_3$, $e_4 \leq e_2$, $e_5 \not\leq e_6$, $e_2 \leq e_5$ and $e_6 \leq e_1$. By transitivity, $e_6 \leq e_3$ and $e_4 \leq e_5$. If $v_1 = w_1$, then $e_3 = e_5$ and $e_4 \leq e_5 = e_3$, a contradiction. Then, $v_1 \neq w_1$ and w_1 is inside C_{e_3}, since $e_6 \leq e_3$. By planarity, $rw_1 = e_5$ is also inside C_{e_3}. Then $e_5 \leq e_3$ and then $e_4 \leq e_5 \leq e_3$, a contradiction since $e_4 \not\leq e_3$.

We then conclude that $e_1 \preceq e_2$ and $e_2 \preceq e_1$ implies that $e_1 \leq e_2$ and $e_2 \leq e_1$, and consequently, $e_1 = e_2$, proving that $\preceq$ is antisymmetric.

We claim that $\preceq$ is transitive. To prove this, suppose that $e_0 \preceq e_1$ and $e_1 \preceq e_2$. If $e_0 \leq e_1$ and $e_1 \leq e_2$, then by transitivity $e_0 \leq e_2$ and consequently $e_0 \preceq e_2$. So, assume that $e_0 \leq e_1$ and $e_1 \not\leq e_2$. Since $e_1 \preceq e_2$, we have by the definition of $\preceq$ that there exist two overstepping edges $e_3 = rv_1$ and $e_4 = v_1v_2$ such that v_1 and v_2 are leaves, $e_4 \not\leq e_3$, $e_1 \leq e_3$ and $e_4 \leq e_2$. By transitivity $e_0 \leq e_3$ and then e_3 and e_4 also satisfy the condition to conclude that $e_0 \preceq e_2$.

Now assume that $e_0 \not\leq e_1$ and $e_1 \not\leq e_2$. Since $e_0 \preceq e_1$ and $e_1 \preceq e_2$, we have by the definition of $\preceq$ that there exist four overstepping edges $e_3 = rv_1$, $e_4 = v_1v_2$, $e_5 = rw_1$ and $e_6 = w_1w_2$ such that v_1, v_2, w_1, w_2 are leaves, $e_4 \not\leq e_3$, $e_0 \leq e_3$, $e_4 \leq e_1$, $e_5 \not\leq e_6$, $e_1 \leq e_5$ and $e_6 \leq e_2$. By transitivity, $e_4 \leq e_5$. If $v_1 = w_1$, then $e_3 = e_5$ and $e_4 \leq e_5 = e_3$, a contradiction. Then, $v_1 \neq w_1$ and v_1 is inside C_{e_5}, since $e_4 \leq e_5$. By planarity, $rv_1 = e_3$ is also inside C_{e_5}. Then $e_3 \leq e_5$ and then $e_0 \leq e_5$. Thus, e_5 and e_6 also satisfy the condition to conclude that $e_0 \preceq e_2$. In other words, $\preceq$ is transitive. □

Lemma 7 *Let x be a vertex of G. If $d_T(x) = 1$, then $d_G(x) \geq 4$.*

Proof. Suppose for a contradiction that $d_T(x) = 1$ and $d_G(x) \leq 3$. By minimality of (G, T), there is a $(G - x, T - x)$-colouring c. At x, at most 3 colours are forbidden by its neighbour in T and at most 2 colours are forbidden by its two other neighbours. So one colour of Z_6 is still available to colour the vertex x. Hence one can extend c to (G, T), a contradiction. □

3.1 Minimal overstepping edges

Lemma 8 *Let $e = uv$ be a minimal overstepping edge. Then there are at most two vertices inside C_e. Moreover if there are two, then they are adjacent in T and one of them is a twig and the other is a leaf.*

Proof. Since G is triangulated, uv is incident to two triangular faces, one of which, say F, is included in C_e^{int}. Let w be the third vertex incident to F. Let P be the path joining u to v

in T and Q be the path joining w to P in T. Since T has diameter 4 and r is on the outer face, then Q has length at most 2.

Then C_e^{int} is divided into at most three regions: F, C_{uw}^{int} and C_{vw}^{int} (the region C_{uw}^{int} or C_{vw}^{int} may not exist if $uw \in E(T)$ or $vw \in E(T)$ respectively). As F is a face, its interior is empty, and there are no vertices inside C_{uw}^{int} and C_{vw}^{int} because uw and vw are not overstepping since e is minimal. Hence the only possible vertices inside C_e are those of Q. Therefore there are at most two vertices inside C_e as Q has length at most 2.

Furthermore, if there are two vertices inside C_e, they must be adjacent as they are in Q. In addition, since r is on the outer face, none of these vertices is the root and thus one of them is a twig and the other is a leaf. □

Lemma 9 *No minimal overstepping edge joins two leaves adjacent to a same twig.*

Proof. Suppose for a contradiction that an edge $e = uv$ joins two leaves adjacent to a same twig t. Then $C_e = tuvt$. The root r is not in C_e^{int} as it is on the outer face. So by Lemma 8 and because G is triangulated, C_e^{int} is a K_4 and there is a unique vertex x inside C_e. Hence, x contradicts Lemma 7. □

Lemma 10 *No minimal overstepping edge joins two twigs.*

Proof. Suppose for a contradiction that two twigs s and t are joined by a minimal edge e. Then $C_e = rstr$. If there is a unique vertex u inside C_e^{int}, then u contradicts Lemma 7. So by Lemma 8, we may assume that the interior of C_e contains two adjacent vertices u_1 and u_2 and that u_1 is a twig and u_2 a leaf. By minimality of (G, T), there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c. Set $c(u_2) = 2$ and choose $c(u_1)$ in $Z_6 \setminus \{1, 2, 3, c(s), c(t)\}$. This yields a (G, T)-colouring, a contradiction. □

Lemma 11 *No minimal overstepping edge joins the root and a leaf.*

Proof. Suppose for a contradiction that a minimal edge e joins the root r and a leaf v. Let t be the twig adjacent to v.

Suppose there is a unique vertex u inside C_e. Then this vertex has only 3 neighbours, and $d_T(u) = 1$. This contradicts Lemma 7. Hence by Lemma 8, we may assume that there are two adjacent vertices u_1 and u_2 inside C_e. Without loss of generality, u_2 is a leaf and u_1 is a twig. By Lemma 7, $d_G(u_2) \geq 4$, so $N_G(u_2) = \{u_1, r, v, t\}$. By minimality of G, there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c. Let $c(u_2)$ be a colour in $\{2, 3\} \setminus \{c(v), c(t)\}$. (Such a colour exists because $|c(v) - c(t)| \geq 2$.) Now by planarity, u_1 has at most one neighbour x in $\{v, t\}$ as ru_2 is an edge. The set of forbidden colours in u_1 is $I = [1] \cup [c(u_2)] \cup \{c(x)\}$ which has cardinality at most 5 by the choice of $c(u_2)$. Hence assigning to u_1 a colour $c(u_1)$ in $Z_6 \setminus I$, we obtain a (G, T)-colouring, a contradiction. □

Lemma 12 *No minimal overstepping edge joins a leaf and a twig.*

Proof. Suppose for a contradiction that a minimal overstepping edge $e = sv$ joins a twig s and a leaf v. Then $C_e = svtrs$. By Lemma 8 there are at most two vertices inside C_e.

Suppose that there is a unique vertex u inside C_e. As $d_T(u) = 1$, by Lemma 7, $d_G(u) \geq 4$. So $N_G(u) = \{r, s, t, v\}$. Note that rv or st is not an edge, by planarity. Then, removing u and contracting rv or st, we find by the minimality of G a $(G - u, T - u)$-colouring c such that $c(v) = 1$ or $c(s) = c(t)$. Since the set of forbidden colours for u has at most 5 colours, one can extend c into a (G, T)-colouring, a contradiction.

Hence by Lemma 8, inside C_e there are a twig u_1 and leaf u_2 which are adjacent in T. As $d_T(u_2) = 1$, $d_G(u_2) \geq 4$ by Lemma 7.

- Suppose first that r is not adjacent to u_2. By Lemma 7, $d_G(u_2) \geq 4$. So $N_G(u_2) = \{u_1, s, t, v\}$.

  Hence u_1 is not adjacent to v by planarity. By minimality of (G, T), there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c. Assign to u_2 a colour $c(u_2)$ in $\{1, 2\} \setminus c(v)$. Observe that it is valid since s and t are not coloured in $\{1, 2\}$. Then the set of forbidden colours in u_1 is included in $\{1, 2, 3, c(s), c(t)\}$ and so has cardinality at most 5. Hence one can extend c into a (G, T)-colouring a contradiction.

- Suppose now that r is adjacent to u_2.

  By planarity, u_1 is adjacent to at most one vertex w in $\{s, t\}$. By minimality of (G, T), there is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c.

  If $c(v) \neq 2$, then set $c(u_2) = 2$. This it is valid since s and t are not coloured 2. Then the set of forbidden colours in u_1 is included in $\{1, 2, 3, c(v), c(w)\}$ and so has cardinality at most 5. Hence one can extend c into a (G, T)-colouring a contradiction. Hence we may assume that $c(v) = 2$.

  If no neighbour of u_2 is coloured 6, then set $c(u_2) = 6$. The set of forbidden colours in u_1 is then $\{1, 2, 5, 6, c(w)\}$ and so one can extend c into a (G, T)-colouring a contradiction. Hence we may assume that a neighbour y of u_2 is coloured 6.

  If no neighbour of u_2 is coloured 3, then set $c(u_2) = 3$. The set of forbidden colours in u_1 is then $\{1, 2, 3, 4, c(w)\}$ and so one can extend c into a (G, T)-colouring a contradiction. Hence we may assume that a neighbour y of u_2 is coloured 3. But this neighbour cannot be t since $c(v) = 2$. Thus $c(s) = 3$ and $c(t) = 6$.

  If $w = s$, that is if u_1 is not adjacent to t, then setting $c(u_1) = 6$ and $c(u_2) = 4$ yields a (G, T)-colouring, a contradiction.

  If $w = t$, then setting $c(u_1) = 3$ and $c(u_2) = 5$ yields a (G, T)-colouring, a contradiction.

□

Lemma 13 *If e is a minimal overstepping edge joining two leaves, then there is one vertex inside C_e.*

Proof. Let $e = v_1 v_2$ and for $i = 1, 2$, let t_i be the twig adjacent to v_i. By Lemma 9, $t_1 \neq t_2$. Since e is minimal and G is triangulated, $u_2 v_1, u_2 v_1 \in E(G)$.

Suppose for a contradiction that more than one vertex is inside C_e. Then, by Lemma 8, inside C_e, there are a twig u_1 and a leaf u_2 which are adjacent in T. Moreover, by Lemma 7, $d_G(u_2) \geq 4$ and so $d_G(u_1) \leq 5$.

Let us first suppose that ru_2 is not an edge. By symmetry, we may assume that u_1v_1 is not an edge. Set $G' = (G - \{u_1, u_2\}) \cup \{rv_1, rv_2\}$. By minimality of (G, T), there is a $(G', T - \{u_1, u_2\})$-colouring, which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c such that $c(v_1) \neq 1$ and $c(v_2) \neq 1$. Then setting $c(u_2) = 1$ and colouring u_1 with a colour in $Z_6 \setminus \{1, 2, c(t_1), c(t_2), c(v_2)\}$, we obtain a (G, T)-colouring, a contradiction. Hence we may assume that $ru_2 \in E(G)$. Then, since e is minimal, u_1v_1 is not an edge. By symmetry, we may assume that ru_2 is inside the cycle $rt_1v_1u_2u_1r$. Thus $N(u_1) \subset \{r, t_2, v_2, u_2\}$.

Assume now that rv_1 is not an edge. Let (G', T') be the graph pair obtained from $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ by identifying r and v_1. By minimality of (G, T), there is a (G', T')-colouring which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c such that $c(v_1) = c(r) = 1$. If $c(v_2) \neq 2$, then setting $c(u_2) = 2$ and colouring u_1 with a colour in $Z_6 \setminus \{1, 2, 3, c(t_2), c(v_2)\}$, we obtain a (G, T)-colouring, a contradiction. If $c(v_2) = 2$, then $c(t_2) \geq 4$. If $c(t_1) \neq 3$, then colour u_2 with 3 and u_1 with some colour in $\{5, 6\} \setminus \{c(t_2)\}$; otherwise, colour u_1 with 3 and u_2 with a colour in $\{5, 6\} \setminus \{c(t_2)\}$. In both cases, we obtain a (G, T)-colouring, a contradiction. Hence we may assume that $rv_1 \in E(G)$.

Assume that rv_2 is not an edge. Let (G', T') be the graph pair obtained from $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ by identifying r and v_2. By minimality of (G, T), there is a (G', T')-colouring which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c such that $c(v_2) = c(r) = 1$. If there is a colour $\alpha \in \{2, 3, 6\}$ which does not appear on the neighbourhood of u_2, then setting $c(u_2) = \alpha$ and colouring u_1 with a colour in $Z_6 \setminus (\{1, 2, c(t_2)\}) \cup [\alpha]$, we obtain a (G, T)-colouring, a contradiction. So all the colours of $\{2, 3, 6\}$ appear on the neighbourhood of u_2. Necessarily, in this case, u_2 is adjacent to v_1, t_1 and t_2 and $c(v_1) = 2$, $c(t_1) = 6$ and $c(t_2) = 3$. Then setting $c(u_2) = 4$ and $c(u_1) = 6$, we obtain a (G, T)-colouring, a contradiction. Hence we may assume that $rv_2 \in E(G)$.

We now distinguish several cases depending on the position of rv_1 and rv_2 regarding C_e.

1. Assume first that rv_1 and rv_2 are in C_e^{ext}. Then t_1t_2 is not an edge by planarity.

   Let (G', T') be the graph pair obtained from $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ by identifying t_1 and t_2. By minimality of (G, T), there is a (G', T')-colouring which is a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c such that $c(t_1) = c(t_2) = \alpha$. If $2 \notin \{c(v_1), c(v_2)\}$, then setting $c(u_2) = 2$ and colouring u_1 with a colour in $Z_6 \setminus \{1, 2, 3, \alpha, c(v_2)\}$, we obtain a (G, T)-colouring, a contradiction. Hence $2 \in \{c(v_1), c(v_2)\}$, so $\alpha \geq 4$.

   If $\{c(v_1), c(v_2)\} \neq \{2, 3\}$, then setting $c(u_2) = 3$ and colouring u_1 with a colour in $\{5, 6\} \setminus \{\alpha, c(v_2)\}$, we obtain a (G, T)-colouring, a contradiction. Hence $\{c(v_1), c(v_2)\} = \{2, 3\}$, so $\alpha \geq 5$.

   If $c(v_2) \neq 3$ or $u_1v_2 \notin E(G)$, then setting $c(u_1) = 3$ and and colouring u_2 with a colour in $\{5, 6\} \setminus \{\alpha\}$, we obtain a (G, T)-colouring, a contradiction. Hence $c(v_2) = 3$ and $u_1v_2 \in E(G)$. By planarity, this implies that u_2t_2 is not an edge.

   Observe that at least one of the two edges rv_1 and rv_2 is not overstepping otherwise one of them would be smaller than e in the order $\preceq$.

   If rv_1 is not overstepping, then the interior of rt_1v_1 is empty. Hence $N_G(t_1) = \{r, v_1, u_2\}$. Setting $c(u_1) = 4$, $c(u_2) = 6$ and recolouring t_1 with 5, we obtain a (G, T)-colouring, a contradiction.

   If rv_2 is not overstepping, then the interior of rt_2v_2 is empty. Hence $N_G(t_2) = \{r, u_1, v_2\}$. Setting $c(u_1) = 6$, $c(u_2) = 4$ and recolouring t_2 with 5, we obtain a (G, T)-colouring, a contradiction.

2. Assume that rv_1 and rv_2 are in C_e^{int}. Then $N_G(u_1) = \{r, u_2, v_2\}$. By minimality of (G, T), there is a $(G-\{u_1, u_2\}, T-\{u_1, u_2\})$-colouring c. Colour u_2 with a colour $c(u_2)$ in $\{2, 3, 6\} \setminus \{c(v_1), c(v_2)\}$. Then the set of forbidden colours in u_1 is $\{1, 2, c(v_2)\} \cup [c(u_2)]$ which has cardinality at most 5 because $\{1, 2\} \cup [c(u_2)]$ has cardinality at most 4. Hence one can extend c into a (G, T)-colouring, a contradiction.

3. Assume that rv_1 is in C_e^{int} and rv_2 is in C_e^{ext}.

   Assume that $d_G(u_2) = 5$, so $N_G(u_2) = \{r, u_1, v_1, v_2, t_2\}$ and $N_G(u_1) = \{r, t_2, u_2\}$. By minimality of (G, T), there is a $(G-\{u_1, u_2\}, T-\{u_1, u_2\})$-colouring c. If one can colour u_2 with a colour in $\{2, 3, 6\}$, then $\{1, 2\} \cup [c(u_2)]$ has cardinality at most 4 and so at most 5 colours are forbidden for u_1. Hence one can extend c into a (G, T)-colouring, a contradiction. So we may assume $\{c(t_2), c(v_1), c(v_2)\} = \{2, 3, 6\}$. If $c(t_2) = 6$, then setting $c(u_1) = 3$ and $c(u_2) = 5$, we obtain a (G, T)-colouring, a contradiction. If $c(t_2) \neq 6$, then setting $c(u_1) = 6$ and $c(u_2) = 4$, we obtain a (G, T)-colouring, a contradiction.

   Henceforth we may assume that $d_G(u_2) = 4$, so $N_G(u_2) = \{r, u_1, v_1, v_2\}$ and $N_G(u_1) = \{r, t_2, v_2, u_2\}$.

   If $\{c(v_1), c(v_2)\} \neq \{2, 3\}$, then one can colour u_2 with a colour in $\{2, 3\}$ and u_1 with a colour in $\{5, 6\} \setminus \{c(t_2), c(v_2)\}$ to obtain a (G, T)-colouring, a contradiction.

   If $\{c(v_1), c(v_2)\} = \{2, 3\}$, then colouring u_1 with a colour $c(u_1)$ in $\{4, 6\} \setminus \{c(t_2)\}$ and u_2 with the colour in $\{4, 6\} \setminus \{c(u_1)\}$, we obtain a (G, T)-colouring, a contradiction.

4. Assume rv_2 is in C_e^{int} and rv_1 is in C_e^{ext}. Then $N_G(u_1) = \{r, u_2, v_2\}$ and $N_G(u_2) = \{r, u_1, t_1, v_1, v_2\}$. By minimality of (G, T), there exists a $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$-colouring c.

   If $c(v_2) = 2$, then colouring u_2 with a colour $c(u_2)$ in $Z_6 \setminus \{1, 2, c(t_1), c(v_1)\}$ and u_1 with a colour in $\{3, 4, 5, 6\} \setminus [c(u_2)]$, we obtain a (G, T)-colouring, a contradiction. So we may assume that $c(v_2) \neq 2$.

   If one can colour u_2 with a colour in $\{2, 3, 6\}$, then $\{1, 2\} \cup [c(u_2)]$ has cardinality at most 4 and so at most 5 colours are forbidden in u_1. Hence one can extend c into a (G, T)-colouring, a contradiction.

   So we may assume $\{c(t_1), c(v_1), c(v_2)\} = \{2, 3, 6\}$. Necessarily, $c(v_1) = 2$, $c(v_2) = 3$ and $c(t_1) = 6$. Setting $c(u_1) = 6$ and $c(u_2) = 4$, we obtain a (G, T)-colouring, a contradiction.

$\square$

Lemma 14 *If e is a minimal overstepping edge, then C_e^{int} is one of the graphs depicted in Figure 2. In addition, if $C_e^{int} = A_1$, then $rv_1 \in E(G)$.*

Proof. Let e be a minimal edge. According to the previous lemmas, it has to join two leaves v_1 and v_2 and there is a unique vertex u inside C_e. For $i = 1, 2$, let t_i be the twig adjacent to v_i. By Lemma 9, $t_1 \neq t_2$.

- Assume first that u is a twig.

  If $d_G(u) \leq 4$, then consider a $(G-u, T-u)$-colouring c, which exists by minimality of (G,T). In u, there are at most 5 colours forbidden as r is coloured 1, and thus forbids only two colours. Hence, one can extend c into a (G,T)-colouring, a contradiction.

  So we may assume that $d_G(u) \geq 5$, and thus $N_G(u) = \{r, t_1, t_2, v_1, v_2\}$.

  If rv_1 is not an edge, then let (G', T') be the pair obtained from $(G-u, T-u)$ by identifying r and v_1. By minimality of (G,T), there is a (G',T')-colouring, which is a $(G-u, T-u)$-colouring such that $c(v_1) = c(r) = 1$. Then the set of forbidden colours in u is included in $\{1, 2, c(t_1), c(t_2), c(v_2)\}$ and so has cardinality at most 5. Hence one can extend c into a (G,T)-colouring, a contradiction.

  Hence we may assume that rv_1 is an edge. This edge must be in C_e^{ext} by planarity of G. Thus t_1t_2 is not an edge of G. Let (G', T') be the pair obtained from $(G-u, T-u)$ by identifying t_1 and t_2. By minimality of (G,T), there is a (G',T')-colouring c which is a $(G-u, T-u)$-colouring such that $c(t_1) = c(t_2)$. Then the set of forbidden colours in u is included in $\{1, 2, c(t_1), c(v_1), c(v_2)\}$ and so has cardinality at most 5. Hence one can extend c into a (G,T)-colouring, a contradiction.

- Assume now that u is a leaf. By symmetry, we may assume that u is adjacent to t_1. By Lemma 7 and since G is triangulated, C_e^{int} is one of the graphs A_1, A_2 or A_3.

Assume now that $C_e^{int} = A_1$ and $rv_1 \notin E(G)$. Let (G', T') be the pair obtained from $(G-u, T-u)$ by identifying r and v_1. By minimality of (G,T), there is a (G',T')-colouring which is a $(G-u, T-u)$-colouring c such that $c(v_1) = c(r) = 1$. Then the set of forbidden colours in u is included in $\{1, c(v_2)\} \cup [c(t_1)]$ and so has cardinality at most 5. Hence one can extend c into a (G,T)-colouring, a contradiction. □

3.2 Penultimate edges

Lemma 15 *Let f be an edge which is the successor of a minimal edge e. If e is the unique predecessor of f, then C_f^{int} is one of the graphs depicted in Figure 3, and $rv_1 \in E(G)$. Morever, if $C_f^{int} = B_2$, $rv_3 \notin E(G)$.*

Proof. Let e' be the third edge of the triangle bounded by f and e in C_f^{int}. Suppose, by way of contradiction, that e is the unique predecessor of f. Then e' is not overstepping. So all the vertices inside C_f are in C_e^{int}. By Lemma 14, C_e^{int} is one of the graphs A_1, A_2 or A_3.

One of the endvertices of f must be v_1 and v_2 (as defined for A_i). We now distinguish many cases depending on C_e^{int} and the possible endvertices of f.

1. Assume that C_e^{int} is A_1.

1.1. Assume $f = rv_1$. Then the 4-cycle $rt_2v_2v_1$ has no chord, because rv_2 is in C_e^{int} and v_1t_2 is not an edge since f is the succesor of e. This contradicts the fact that G is triangulated.

1.2 Observe that $f = t_1v_2$ is impossible since rv_1 is an edge. Assume that $f = t_2v_1$. Let $G' = (G - \{u, v_2\}) \cup t_1t_2$. By minimality of (G,T), there exists a $(G', T - \{u, v_2\})$-colouring which is a $(G-\{u,v_2\},T-\{u,v_2\})$-colouring c such that $c(t_1)\neq c(t_2)$. If $c(t_1)=6$, then one can greeedily extend c to v_2 and then u to get a (G,T)-colouring, a contradiction. If $c(t_1)\neq 6$, then colouring v_2 with a colour in $\{c(t_1)-1,c(t_1)+1\}\setminus[c(t_2)]$ and u with a colour in $Z_6\setminus(\{1,c(v_1)\}\cup[c(t_1)])$ we obtain a (G,T)-colouring, a contradiction.

1.3. Assume that $f=v_1t_3$ with t_3 a twig distinct from t_2. Since rv_1 is an edge, t_1t_3 is not an edge. Let G' be the graph pair obtained from $(G-\{t_2,v_2\},T-\{t_2,v_2\})$. If one can colour v_2 with a colour $c(v_2)$ in $\{2,3,6\}$, then $\{1,2\}\cup[c(v_2)]$ has cardinality at most 4 and so at most 5 colours are forbidden in t_2. Hence one can extend c into a (G,T)-colouring , a contradiction. So we may assume that $\{c(u),c(v_1),c(t_3\}=\{2,3,6\}$. If $c(t_3)=3$, set $c(v_2)=4$ and $c(t_2)=6$. If $c(t_3)=6$, set $c(v_2)=5$ and $c(t_2)=3$. In both cases, we obtain a (G,T)-colouring, a contradiction.

1.4. Assume that $f=v_2t_3$ with t_3 a twig distinct from t_1. By minimality of (G,T), there exists a $(G-\{u,t_1,v_1\},T-\{u,t_1,v_1\})$-colouring c. Setting $c(t_1)=6$ and choosing $c(v_1)$ in $Z_6\setminus\{1,5,6,c(t_3),c(v_2)\}$ and $c(u)$ in $Z_6\setminus\{1,5,6,c(v_1),c(v_2)\}$, we get a (G,T)-colouring, a contradiction.

1.5. f cannot be v_2v_3 with v_3 a leaf adjacent to t_1 because rv_1 is an edge.

1.6 Assume that $f=v_1v_3$ with v_3 a leaf adjacent to t_2. Then $C_e^{int}=B_1$. By Lemma 14, $rv_1\in E(G)$.

1.7. Assume $f=v_2v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1,t_2\}$. Then $v_1v_3\in E(G)$ and either $rv_3\in E(G)$ or $t_3v_1\in E(G)$. Since rv_1 is an edge, we have that $N(t_1)=\{r,u,v_1\}$. By minimality of G, there exists a $(G-\{u,t_1,v_1\},T-\{u,t_1,v_1\})$-colouring c.

If $\{c(t_3),c(v_3),c(v_2)\}\neq\{2,3,4\}$, then setting $c(t_1)=6$ and choosing $c(v_1)$ in $\{2,3,4\}\setminus\{c(t_3),c(v_3),c(v_2)\}$ and $c(u)$ in $\{2,3,4\}\setminus\{c(v_1),c(v_2)\}$, we obtain a (G,T)-colouring, a contradiction. Hence $\{c(t_3),c(v_3),c(v_2)\}=\{2,3,4\}$, and so $c(t_3)=4$, $c(v_3)=2$ and $c(v_2)=3$. Then setting $c(t_1)=3$, $c(u)=5$ and $c(v_1)=6$ yields a (G,T)-colouring, a contradiction.

1.8. Assume $f=v_1v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1,t_2\}$. Since $rv_1\in E(G)$, then $t_1t_3\notin E(G)$.

Assume first that $rv_3\in C_f^{int}$. By minimality of (G,T), there exists a $(G-\{t_2,v_2\},T-\{t_2,v_2\})$-colouring c. One can choose a colour $c(v_2)$ in $Z_6\setminus\{1,c(u),c(v_1),c(v_3)\}$ such that $I=[c(v_2)]\cup\{1,2,c(v_3)\}\neq Z_6$. Then choosing $c(t_2)\in Z_6\setminus I$, we obtain a (G,T)-colouring, a contradiction.

Hence we may assume that rv_3 is not in C_f^{int}. Let (G',T') be the graph obtained from $(G-\{u,t_2,v_2\},T-\{u,t_2,v_2\})$ by identifying t_1 and t_3. By minimality of (G,T), there exists a (G',T')-colouring which is a $(G-\{u,t_2,v_2\},T-\{u,t_2,v_2\})$-colouring c such that $c(t_1)=c(t_3)$. If $c(t_1)\neq 6$, then one can choose a colour $c(v_2)\in\{c(t_1)-1,c(t_1)+1\}$ such that $I=[c(v_2)]\cup\{1,2,c(v_3)\}\neq Z_6$. Then choosing $c(t_2)\in Z_6\setminus I$ and $c(u)$ in $Z_6\setminus([c(t_1)]\cup\{1,c(v_1)\}$, we obtain a (G,T)-colouring, a contradiction. Hence we may suppose that $c(t_1)=6$. If $v_2t_3\notin E(G)$, then setting $c(v_2)=6$ and choosing $c(t_2)\in\{3,4\}\setminus c(v_3)$ and $c(u)$ in $Z_6\setminus\{1,5,6,c(v_1)\}$ yields a (G,T)-colouring, a contradiction.

If $v_2t_3 \in E(G)$, then setting $c(v_2) = 5$, $c(t_2) = 3$ and choosing $c(u)$ in $Z_6 \setminus \{1, 5, 6, c(v_1)\}$ yields a (G, T)-colouring, a contradiction.

2. Assume that C_e^{int} is A_2.

2.1. Assume $f = rv_1$. Since f is the successor of e, then v_1t_2 is not an edge and so $rv_2 \in E(G)$ because G is triangulated. By minimality of G, there exists a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c. Setting $c(u) = 1$, one can then extend c greedily to t_2 and v_2 to get a (G, T)-colouring, a contradiction.

2.2. Assume that $f = rv_2$. By minimality of (G, T), there is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring. Setting $c(u) = 1$, one can then extend c greedily to t_1 and v_1 to get a (G, T)-colouring, a contradiction.

2.3 Assume that $f = t_1v_2$. Since f is the sucessor of e, the cycle $t_1v_1v_2$ is empty, an so v_1 contradicts Lemma 7. Similarly, if $f = t_2v_1$, then v_2 contradicts Lemma 7.

2.4. Assume that $f = v_1t_3$ with t_3 a twig distinct from t_2. Since f is the sucessor of e, t_2v_1 is not an edge. Then either rv_2 is an edge or t_2t_3 is an edge. Set $G' = (G - \{u, t_2, v_2\}) \cup rv_1$. By minimality of (G, T), there is a $(G', T - \{u, t_2, v_2\})$-colouring c which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c such that $c(v_1) \neq c(r) = 1$. Set $c(u) = 1$.

If $c(v_1) \neq 2$, then setting $c(v_2) = 2$ and colouring t_2 with a colour in $Z_6 \setminus \{1, 2, 3, c(t_1), c(t_3)\}$, we obtain a (G, T)-colouring, a contradiction. So $c(v_1) = 2$ and thus $c(t_1) \geq 4$.

If $c(t_3) \neq 3$, then setting $c(t_2) = 3$ and choosing $c(v_2)$ in $\{5, 6\} \setminus \{c(t_3)\}$, we obtain a (G, T)-colouring, a contradiction. So $c(t_3) = 3$.

Choosing $c(t_2)$ in $\{4, 6\} \setminus \{c(t_1)\}$ and $c(v_2)$ in $\{4, 6\} \setminus \{c(t_2)\}$, we obtain a (G, T)-colouring, a contradiction.

2.5. Assume that $f = v_2t_3$ with t_3 a twig distinct from t_1. Then either rv_1 is an edge or t_1t_3 is an edge. Set $G' = (G - \{u, t_1, v_1\}) \cup rv_2$. By minimality of (G, T), there exists a $(G', T - \{u, t_1, v_1\})$-colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring c such that $c(v_2) \neq c(r) = 1$. Set $c(u) = 1$.

If $c(v_2) \neq 2$, then setting $c(v_1) = 2$ and colouring t_1 with a colour in $Z_6 \setminus \{1, 2, 3, c(t_2), c(t_3)\}$, we obtain a (G, T)-colouring, a contradiction. So $c(v_2) = 2$ and thus $c(t_2) \geq 4$.

If $c(t_3) \neq 3$, then setting $c(t_1) = 3$ and choosing $c(v_1)$ in $\{5, 6\} \setminus \{c(t_3)\}$, we obtain a (G, T)-colouring, a contradiction. So $c(t_3) = 3$.

Choosing $c(t_1)$ in $\{4, 6\} \setminus \{c(t_2)\}$ and $c(v_1)$ in $\{4, 6\} \setminus \{c(t_1)\}$, we obtain a (G, T)-colouring, a contradiction.

2.6. Assume that $f = v_2v_3$ with v_3 a leaf adjacent to t_1. Since f is the successor of e, then t_1v_2 is not inside $v_3t_1v_1v_2$ and so $v_1v_3 \in E(G)$. Set $G' = (G - \{u, v_1\}) \cup t_2v_3$. By minimality of (G, T), there is a $(G', T - \{u, v_1\})$-colouring which is $(G - \{u, v_1\}, T - \{u, v_1\})$-colouring c such that $c(t_2) \neq c(v_3)$. Setting $c(u) = c(v_3)$ and colouring v_1 with a colour in $Z_6 \setminus (\{c(u), c(v_2)\} \cup [c(t_1)])$, we obtain a (G, T)-colouring, a contradiction.

2.7. Assume that $f = v_1v_3$ with v_3 a leaf adjacent to t_2. Since f is the successor of e, then t_2v_1 is not inside $v_2t_2v_3v_1$ and so $v_2v_3 \in E(G)$. Set $G' = (G-\{u, v_2\}) \cup t_1v_3$. By minimality of (G, T), there is a $(G', T-\{u, v_2\})$-colouring, which is a $(G-\{u, v_2\}, T-\{u, v_2\})$-colouring c such that $c(t_1) \neq c(v_3)$. If $c(t_2) \in [c(t_1)]$, then one can also extend c greedily to v_2 and then u to obtain a (G, T)-colouring, a contradiction. Hence $|c(t_1)-c(t_2)| \geq 2$. Thus one can colour v_2 with $c(t_1)$ and then colour u with a colour in $Z_6 \setminus ([c(t_1)] \cup \{c(t_2), c(v_1)\})$. This yields a (G, T)-colouring, a contradiction.

2.8. Assume $f = v_2v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$.

Suppose first that $rv_1 \notin E(G)$. By minimality of (G, T), there is a $(G - \{u, t_1, v_1\} \cup \{rv_3, rv_2\}, T-\{u, t_1, v_1\})$-colouring which is a $(G-\{u, t_1, v_1\}, T-\{u, t_1, v_1\})$-colouring c such that $c(v_2) \neq 1$ and $c(v_3) \neq 1$. Colour v_1 with 1 and let $L(t_1) \supseteq Z_6 \setminus \{1, 2, c(v_3), c(t_3), c(t_2)\}$ and $L(u) = Z_6 \setminus \{1, c(t_2), c(v_2)\}$ be the list of colours available for t_1 and u, respectively. Note that there is at most one colour α in Z_6 such that $L(u) \setminus [\alpha] = \emptyset$. Thus, if there exists β in $L(t_1) \setminus \{\alpha\}$ if such α exists, or in $L(t_1)$ otherwise, then we can colour t_1 with β and u with a colour in $L(u) \setminus [\beta]$ to obtain a (G, T)-colouring, a contradiction. So we may assume that no such β exists, that is $L(t_1) = \{\alpha\}$ and $L(u) \setminus [\alpha] = \emptyset$. Since $|c(v_2) - c(t_2)| \geq 2$, necessarily $\alpha = 4$, $L(t_1) = \{4\}$, $c(t_2) = 6$, $c(v_2) = 2$, $\{c(v_3), c(t_3)\} = \{3, 5\}$ and $v_3, t_3 \in N(t_1)$. Then, recolouring v_1 with 6 and colouring t_1 with 4 and u with 1 yields a (G, T)-colouring, a contradiction.

Suppose now that $rv_1 \in E(G)$. Then there is no vertex inside rt_1v_1r. By minimality of (G, T), there is $(G - u, T - u)$-colouring c. If $c(v_2) \neq 1$, then we can colour u with 1; so, suppose otherwise. If there is no colour available for u to extend c, then $F_c = \{1, c(t_2), c(v_1)\} \cup [c(t_1)]$ is equal to Z_6; thus, $c(t_1) \in \{3, 4, 5\}$. If $c(t_1) = 3$, then $\{c(v_1), c(t_2)\} = \{5, 6\}$. If $c(t_1) = 4$, then $\{c(v_1), c(t_2)\} = \{2, 6\}$. If $c(t_1) = 5$, then $\{c(v_1), c(t_2)\} = \{2, 3\}$. If the colour of t_1 can be changed, we obtain a $(G - u, T - u)$-colouring c' such that $F_{c'} \neq Z_6$ which can be extended in a (G, T)-colouring, a contradiction. Hence, $c(t_1) = i$ is the sole colour in $Z_6 \setminus (\{1, 2, c(t_2)\} \cup [c(v_1)])$. Thus, $c(v_1) \neq 2$ and $(c(v_1), c(t_2)) \neq (6, 5)$. Then, necessarily (*) $c(v_1) = 5$, $c(t_1) = 3$ and $c(t_2) = 6$. If $c(t_3), c(v_3) \neq 3$, then recolour t_1 with 5 and v_1 with 3. Otherwise, if $c(t_3), c(v_3) \neq 6$, then recolour v_1 with 6. Otherwise (i.e., $\{c(t_3), c(v_3)\} = \{3, 6\}$), recolour v_1 with 2. In any case, the resulting colouring c_1 does not satisfy (*). Hence, either $F_{c_1} \neq Z_6$ or t_1 can be recoloured to get a colouring c_1' such that $F_{c_1'} \neq Z_6$. Hence one of c_1, c_1' can be extended in a (G, T)-colouring, a contradiction.

2.9. Assume $f = v_1v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$.

Suppose first that $rv_2 \in E(G)$. Set $G' = (G-\{u, t_2, v_2\}) \cup \{t_1t_3, t_1v_3\}$. By minimality of (G, T), there is a $(G', T-\{u, t_2, v_2\})$-colouring, which is a $(G-\{u, t_2, v_2\}, T-\{u, t_2, v_2\})$-colouring c such that $c(t_1) \neq c(t_3)$ and $c(t_1) \neq c(v_3)$. Set $c(v_2) = c(t_1)$. Then choosing $c(t_2)$ in $\{3, 4, 5, 6\} \setminus [c(t_1)]$ and $c(u)$ in $Z_6 \setminus ([c(t_1)] \cup \{c(t_2), c(v_1)\})$, we obtain a (G, T)-colouring, a contradiction. Hence $rv_2 \notin E(G)$.

Suppose now that $rv_3 \notin E(G)$. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ by identifying v_3 and r. By minimality of (G, T), there is a (G', T')-colouring, which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c such that $c(v_3) = 1$. Set $c(u) = 1$. For t_2, there at least two possible colours, namely the ones not in $\{1, 2, c(t_1), c(t_3)\}$. One of them, say α, is such that $I = [\alpha] \cup \{1, c(v_1), c(t_3)\}$

is not equal to Z_6. Thus, setting $c(t_2) = \alpha$ and choosing $c(v_2)$ in $Z_6 \setminus I$, we obtain a (G, T)-colouring, a contradiction. Hence $rv_3 \in E(G)$.

Assume that rv_3 is inside C_f. Then $t_2t_3 \notin E(G)$. By minimality of (G, T), there is a $(G - \{u, t_2, v_2\} \cup rv_1, T\{u, t_2, v_2\})$-colouring, which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c such that $c(v_1) \neq 1$. Thus, setting $c(v_2) = 1$ and colouring u with a colour in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ and t_1 with a colour in $Z_6 \setminus \{1, 2, c(t_1), c(u), c(v_3)\}$, we obtain a (G, T)-colouring, a contradiction. Hence we may assume that rv_3 is outside C_f.

So, by planarity, $t_1t_3 \notin E(G)$. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ by identifying t_1 and t_3. By minimality of (G, T), there is a (G', T')-colouring, which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c such that $c(t_1) = c(t_3)$. Set $c(u) = c(v_3)$. Let α be a colour of $Z_6 \setminus \{1, 2, c(t_1), c(v_3)\}$ such that $I = [\alpha] \cup \{c(v_1), c(v_3), c(t_3)\}$ is not Z_6. Then setting $c(t_2) = \alpha$ and choosing $c(v_2)$ in $Z_6 \setminus I$, we obtain a (G, T)-colouring, a contradiction.

3. Assume that C_e^{int} is A_3.

3.1. Assume $f = rv_1$. Then rv_2 is an edge. By minimality of G, there exists a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c. Colour u with a colour $c(u)$ in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$. Set $c(t_2) = 6$ if $c(u) \neq 6$ and $c(t_2) = 5$ otherwise. In both cases, at most five colours are forbidden for v_2, and one can extend greedily the colouring into a (G, T)-colouring, a contradiction.

3.2. Assume that $f = rv_2$. Then rv_1 is an edge. By minimality of G, there exists a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring c. Set $c(t_1) = 6$, then colour u with any colour in $Z_6 \setminus \{1, 5, 6, c(t_2), c(v_2)\}$ and v_1 with any colour in $Z_6 \setminus \{1, 5, 6, c(v_2), c(u)\}$. This yields a (G, T)-colouring, a contradiction.

3.3 Assume that $f = t_1v_2$. Since f is the sucessor of e, then the cycle $t_1v_1v_2$ is empty, and so v_1 contradicts Lemma 7. Similarly, if $f = t_2v_1$, then v_2 contradicts Lemma 7.

3.4. Assume that $f = v_1t_3$ with t_3 a twig distinct from t_2.

Assume first that $t_2t_3 \in E(G)$. Then rv_2 is not an edge. Set $G' = (G - \{u, t_2, v_2\}) \cup rv_1$. By minimality of (G, T), there is a $(G', T - \{u, t_2, v_2\})$-colouring which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c such that $c(v_1) \neq c(r) = 1$. Setting $c(v_2) = 1$ and choosing $c(u)$ in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ and $c(t_2)$ in $Z_6 \setminus \{1, 2, c(u), c(t_3)\}$, we get a (G, T)-colouring, a contradiction. So $t_2t_3 \notin E(G)$ and thus $rv_2 \in E(G)$.

By minimality of G, there exists a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c.

Assume that $c(v_1) \neq 2$. If $c(t_1) = 3$, then setting $c(v_2) = 2$ and choosing $c(u)$ in $Z_6 \setminus \{1, 2, 3, 4, c(v_1)\}$ and $c(t_2)$ in $Z_6 \setminus \{1, 2, 3, c(u)\}$ yields a (G, T)-colouring, a contradiction. If $c(t_1) \geq 4$, then setting $c(u) = 2$ and choosing $c(v_2)$ in $Z_6 \setminus \{1, 2, c(v_1), c(t_3)\}$ and $c(t_2)$ in $Z_6 \setminus (\{1, 2\} \cup [c(v_2)])$, we obtain a (G, T)-colouring, a contradiction. Hence $c(v_1) = 2$.

If $c(t_1) \neq 4$, then colouring v_2 with $c(v_2) \in \{4, 6\} \setminus \{c(t_3)\}$, t_2 with $c(t_2) \in \{4, 6\} \setminus \{c(v_2)\}$ and u with $c(u)$ in $\{3, 5\} \setminus [c(t_1)]$, we get a (G, T)-colouring, a contradiction. So $c(t_1) = 4$.

Colouring u with 6, v_2 with $c(v_2) \in \{3, 5\} \setminus \{c(t_3)\}$ and t_2 with $c(t_2)$ in $\{3, 5\} \setminus [c(v_2)]$, we get a (G, T)-colouring, a contradiction.

3.5. Assume that $f = v_2t_3$ with t_3 a twig distinct from t_1.

Assume first that t_1t_3 is an edge. Set $G' = (G - \{u, t_1, v_1\}) \cup rv_2$. By minimality of (G, T), there is a $(G', T - \{u, t_1, v_1\})$-colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring c such that $c(v_2) \neq c(r) = 1$. Set $c(v_1) = 1$. If $c(v_2) \neq 2$, then setting $c(u) = 2$ and assigning to t_1 a colour in $Z_6 \setminus \{1, 2, 3, c(t_3)\}$, we obtain a (G, T)-colouring, a contradiction. So $c(v_2) = 2$ and $c(t_2) \geq 4$. Setting $c(u) = 3$ and assigning to t_1 a colour in $Z_6 \setminus \{1, 2, 3, 4, c(t_3)\}$, we obtain a (G, T)-colouring, a contradiction. Hence t_1t_3 is not an edge.

So rv_1 is an edge. Since e is minimal, then rv_1 is not overstepping and $C^{int}_{rv_1}$ is empty. Let G' be the graph from $G - \{u, t_1, v_1\}$ by adding the edge t_2t_3 if it does not exist. By minimality of (G, T), there is a $(G', T - \{u, t_1, v_1\})$-colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring c such that $c(t_2) \neq c(t_3)$. Set $c(t_1) = 6$. If $c(t_2) \notin \{5, 6\}$, then set $c(v_1) = c(t_2)$ (this is possible because $c(t_3) \neq c(t_2)$), otherwise colour v_1 with any colour in $Z_6 \setminus \{1, 5, 6, c(t_3), c(v_2)\}$. Then colouring u with a colour in $Z_6 \setminus \{1, 5, 6, c(v_1), c(v_2)\}$, we get a (G, T)-colouring, a contradiction.

3.6. Assume that $f = v_2v_3$ with v_3 a leaf adjacent to t_1. Set $G' = (G - \{u, v_1\} \cup \{t_2v_3, rv_3\}$. By minimality of (G, T), there is a $(G', T - \{u, v_1\})$-colouring which is a $(G - \{u, v_1\}, T - \{u, v_1\})$-colouring c such that $c(v_3) \notin \{c(r), c(t_2)$. Setting $c(u) = c(v_3)$ and colouring v_1 with a colour in $Z_6 \setminus (\{c(u), c(v_2)\} \cup [c(t_1)])$, we obtain a (G, T)-colouring, a contradiction.

3.7. Assume that $f = v_1v_3$ with v_3 a leaf adjacent to t_2. Then $C^{int}_f = B_2$.

Assume first that $rv_3 \in E(G)$. Then $t_1t_2 \notin E(G)$. Let (G', T') be the graph pair obtained from $(G - u, T - u)$ by identifying t_1 and t_2. By minimality of (G, T), there is a (G', T')-colouring, which yields a $(G - u, T - u)$-colouring such that $c(t_1) = c(t_2)$. Then setting $c(u) = c(v_3)$, we obtain a (G, T)-colouring, a contradiction. Hence $rv_3 \notin E(G)$.

Now assume that $rv_1 \notin E(G)$. Let (G', T') be the graph pair obtained from $(G - u, T - u)$ by identifying v_1 and r. By minimality of (G, T), there is a (G', T')-colouring, which yields a $(G - u, T - u)$-colouring such that $c(v_1) = 1$. If $c(t_1) = c(t_2)$, then, setting $c(u) = c(v_3)$, we obtain a (G, T)-colouring, a contradiction. If $c(t_1) = 6$ or $c(t_2) \in [c(t_1)]$ or $c(v_2) \in [c(t_1)]$, then colouring u with a colour in $Z_6 \setminus (\{1, c(t_2), c(v_2)\} \cup [c(t_1)])$, we obtain a (G, T)-colouring, a contradiction. So, assume that $c(t_1) \neq 6$ and $c(t_2) \notin [c(t_1)]$ and $c(v_2) \notin [c(t_1)]$. If $c(t_1) = 3$, then $c(t_2), c(v_2) \in \{5, 6\}$, a contradiction. If $c(t_1) = 5$, then $c(t_2), c(v_2) \in \{2, 3\}$, a contradiction. Then, $c(t_1) = 4$, $c(t_2) = 6$ and $c(v_2) = 2$. Recolouring v_2 with a colour in $\{3, 4\} \setminus \{c(v_3)\}$ and setting $c(u) = 2$, we obtain a (G, T)-colouring, a contradiction. Hence $rv_1 \in E(G)$.

3.8. Assume $f = v_2v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$.

Assume first that rv_1 is not an edge. By minimality of (G, T), there is a $(G - \{u, t_1, v_1\} \cup \{rv_2, rv_3\}, T - \{u, t_1, v_1\})$-colouring which is a $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring c such that $c(v_2) \neq 1$ and $c(v_3) \neq 1$. Colour v_1 with 1. Colour t_1 with a colour α in $A = Z_6 \setminus \{1, 2, c(t_3), c(v_3)\}$ such that $[\alpha] \neq Z_6 \setminus \{1, c(t_2), c(v_2)\}$. This is possible since $|A| \geq 2$. Then colouring u with a colour in $Z_6 \setminus (\{1, c(t_2), c(v_2)\} \cup [\alpha])$, we obtain a (G, T)-colouring, a contradiction.

Suppose now that rv_1 is an edge. Then t_1t_3 and t_1v_3 are not edges. By minimality of (G, T), there is a $(G - \{u, t_1, v_1\} \cup \{t_2t_3, t_2v_3\}, T - \{u, t_1, v_1\})$-colouring which is a

$(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$-colouring c such that $c(t_3) \neq c(t_2)$ and $c(v_3) \neq c(t_2)$. Set $c(t_1) = 6$. Let $L(u) = \{2, 3, 4\} \setminus \{t_2, v_2\}$ and let $L(v_1) = \{2, 3, 4\} \setminus \{v_2, t_3, v_3\}$. If $L(v_1)$ is empty, then $c(t_3) = 4$, $c(v_3) = 2$ and $c(v_2) = 3$. In this case, recolouring t_1 with 3, colouring u with a colour in $\{5, 6\} \setminus \{c(t_2)\}$ and colouring v_1 with a colour in $\{5, 6\} \setminus \{c(u)\}$, we obtain a (G, T)-colouring, a contradiction. Hence $L(v_1)$ is not empty. If $|L(u)| > 1$, we can colour v_1 with a colour in $L(v_1)$ and colour u with a colour in $L(u) \setminus \{c(v_1)\}$ to obtain a (G, T)-colouring, a contradiction. Then $|L(u)| = 1$ and consequently $c(t_2) = 4$ and $c(v_2) = 2$. Then colouring u with 3 and colouring v_1 with 4, we obtain a (G, T)-colouring, since $c(t_3)$ and $c(v_3)$ are distinct from $c(t_2) = 4$, a contradiction.

3.9. Assume $f = v_1v_3$ with v_3 a leaf adjacent in T to a twig t_3 not in $\{t_1, t_2\}$.

Suppose first that $rv_2 \notin E(G)$. By minimality of (G, T), there is a $(G - \{t_2, v_2\} \cup \{rv_1, rv_3\}, T - \{t_2, v_2\})$-colouring which is a $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$-colouring c such that $c(v_1) \neq 1$ and $c(v_3) \neq 1$. Setting $c(v_2) = 1$ and choosing $c(t_2)$ in $Z_6 \setminus \{1, 2, c(u), c(t_3), c(v_3)\}$ yields a (G, T)-colouring, a contradiction. Hence $rv_2 \in E(G)$.

Assume that $v_2t_3 \notin E(G)$. By minimality of (G, T), there is a $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$-colouring. We can choose $c(v_2)$ in $Z_6 \setminus \{1, c(u), c(v_1), c(v_3)\}$ such that $I = [c(v_2)] \cup \{1, 2, c(u)\} \neq Z_6$ and $c(t_2) \in Z_6 \setminus I$ to obtain a (G, T)-colouring, a contradiction. Hence $v_2t_3 \in E(G)$.

Now assume that rv_3 is not an edge. Let (G', T') be the graph pair obtained from $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ by identifying v_3 and r. By minimality of (G, T), there is a (G', T')-colouring, which yields a $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$-colouring such that $c(v_3) = 1$. Then colouring v_2 with a colour $\alpha \in Z_6 \setminus \{1, c(u), c(v_1), c(t_3)\}$ such that $[\alpha] \cup \{1, 2, c(u)\} \neq Z_6$ and colouring t_2 with a colour in $Z_6 \setminus ([\alpha] \cup \{1, 2, c(u)\})$, we obtain a (G, T)-colouring, a contradiction.

Hence, rv_3 is an edge and, since v_2t_3 is an edge, t_1t_3 is not an edge by planarity. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ by identifying t_1 and t_3. By minimality of (G, T), there is a (G', T')-colouring, which yields a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring such that $c(t_1) = c(t_3)$. Then setting $c(u) = c(v_3)$, colouring v_2 with a colour $\alpha \in Z_6 \setminus \{1, c(v_1), c(t_3), c(v_3)\}$ such that $[\alpha] \cup \{1, 2, c(u)\} \neq Z_6$ and colouring t_2 with a colour in $Z_6 \setminus ([\alpha] \cup \{1, 2, c(u)\})$, we obtain a (G, T)-colouring, a contradiction.

□

Lemma 16 *Every penultimate edge has a unique predecessor.*

Proof. By contradiction. Suppose that a penultimate edge f has two predecessors e and e'. Then e and e' are ultimate and so minimal. According to Lemma 14, C_e^{int} and $C_{e'}^{int}$ are isomorphic to some of A_1, A_2 or A_3. Let us denote the vertices of C_e^{int} by their names in Figure 2 and the vertices of $C_{e'}^{int}$ by their names in Figure 2 augmented with a prime.

Since f, e and e' are bounding the face incindent to f in C_f^{int}, the edge f is v_1v_2', v_1v_1', v_2v_2' or v_2v_1'. If$f = v_2v_1'$, then swapping the names of e and e', we are left with $f = v_1v_2'$. Hence we may assume that $f \in \{v_1v_2', v_1v_1', v_2v_2'\}$. Note that if $f = v_1v_2'$, then $t_2 = t_1'$ and $v_2 = v_1'$, if $f = v_1v_1'$, then $t_2 = t_2'$ and $v_2 = v_2'$, and if $f = v_2v_2'$, then $t_1 = t_1'$ and $v_1 = v_1'$.

Observe that if C_e^{int} is isomorphic to A_1, then f cannot be v_2v_2' because rv_1 must be an edge that would cross f. Moreover if C_e^{int} and $C_{e'}^{int}$ are both isomorphic to A_1, then f cannot be v_1v_1' since G has no multiple edges. Hence must be in one of the following cases:

- C_e^{int} and $C_{e'}^{int}$ are isomorphic to A_1 and $f = v_1v_2'$.

  By minimality of G, there is a $(G - \{u', t_2, v_2\}, T - \{u', t_2, v_2\})$-colouring c. Colour t_2 with 6. If $\{c(u), c(v_1), c(v_2')\} \neq \{2, 3, 4\}$, then colouring v_2 with a colour in $Z_6 \setminus \{1, 5, 6, c(u), c(v_1), c(v_2')\}$ and colouring u' with a colour in $\{2, 3, 4\} \setminus \{c(v_2), c(v_2')\}$, we obtain a (G, T)-colouring, a contradiction. If $\{c(u), c(v_1), c(v_2')\} = \{2, 3, 4\}$, then recolouring t_2 with 3, and setting $c(v_2) = 5$ and $c(u') = 6$, we obtain a (G, T)-colouring, a contradiction.

- C_e^{int} is isomorphic to A_1, $C_{e'}^{int}$ is isomorphic to A_2 and $f = v_1v_i'$ for $i \in \{1, 2\}$.

  By minimality of (G, T), there is a $(G - \{t_2, v_2, u'\} \cup rv_i', T - \{t_2, v_2, u'\})$-colouring which is a $(G - \{t_2, v_2, u'\}, T - \{t_2, v_2, u'\})$-colouring c such that $c(v_i') \neq 1$. Colouring u' with 1, colouring v_2 with a colour $\alpha \in Z_6 \setminus \{1, c(u), c(v_1), c(v_i')\}$ such that $\{1, 2, c(t_i')\} \cup [\alpha] \neq Z_6$ and colouring t_2 with a colour in $Z_6 \setminus (\{1, 2, c(t_i')\} \cup [\alpha])$, we obtain a $G, T)$-colouring, a contradiction.

- C_e^{int} is isomorphic to A_1, $C_{e'}^{int}$ is isomorphic to A_3 and $f = v_1v_2'$.

  By minimality of (G, T), there is a $(G - \{t_2, v_2, u'\} \cup \{ut_2', v_1t_2'\}, T - \{t_2, v_2, u'\})$-colouring which is a $(G - \{t_2, v_2, u'\}, T - \{t_2, v_2, u'\})$-colouring c such that $c(u) \neq c(t_2')$ and $c(v_1) \neq c(t_2')$. If $\{c(u), c(v_1), c(v_2')\} = \{2, 3, 4\}$, then colour t_2 with 3, colour u' with a colour in $\{5, 6\} \setminus \{c(t_2')\}$ and colour v_2 with a colour in $\{5, 6\} \setminus \{c(u')\}$. If $2 \notin \{c(u), c(v_1), c(v_2')\}$, then set $c(t_2) = 6$, $c(v_2) = 2$ and colour u' with a colour in $\{3, 4\} \setminus \{c(t_2'), c(v_2')\}$. If $4 \notin \{c(u), c(v_1), c(v_2')\}$, then set $c(t_2) = 6$, $c(v_2) = 4$ and colour u' with a colour in $\{2, 3\} \setminus \{c(t_2'), c(v_2')\}$. In any of these cases, we obtain a (G, T)-colouring, a contradiction. So $2, 4 \in \{c(u), c(v_1), c(v_2')\}$ and $3 \notin \{c(u), c(v_1), c(v_2')\}$. Colour t_2 with 6 and v_2 with 3. Notice that $\{c(t_2'), c(v_2')\} \neq \{2, 4\}$, since necessarily $c(t_2') = 4$ and $c(v_2') = 2$, but $c(u), c(v_1) \neq c(t_2') = 4$ contradicts the fact that $2, 4 \in \{c(u), c(v_1), c(v_2')\}$. Then colouring u' with $\{2, 4\} \setminus \{c(t_2'), c(v_2')\}$, we obtain a (G, T)-colouring, a contradiction.

- C_e^{int} is isomorphic to A_1, $C_{e'}^{int}$ is isomorphic to A_3 and $f = v_1v_1'$.

  By minimality of (G, T), there is a $(G - \{u, t_2, v_2\} \cup \{t_1u', t_1v_1'\}, T - \{u, t_2, v_2\})$-colouring which is a $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$-colouring c such that $c(u') \neq c(t_1)$ and $c(v_1') \neq c(t_1)$. Colour u with a colour in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$. If $c(v_1) = 1$ or $c(v_1') = 1$ or $|\{c(u), c(u'), c(v_1), c(v_1')\}| < 4$, then colouring v_2 with a colour $\alpha \in Z_6 \setminus \{1, c(u), c(u'), c(v_1), c(v_1')\}$ such that $[\alpha] \cup \{1, c(u')\} \neq Z_6$ and colouring $c(t_2)$ with a colour in $Z_6 \setminus ([\alpha] \cup \{1, c(u')\})$, we obtain a (G, T)-colouring, a contradiction.

  Then $\{c(u), c(v_1)\} \cap \{c(u'), c(v_1')\} = \emptyset$. This is only possible if $\{c(t_1), c(t_1')\} = \{3, 5\}$, $\{c(t_1), c(t_1')\} = \{3, 6\}$ or $\{c(t_1), c(t_1')\} = \{4, 6\}$. If $c(t_1') = 3$, then $\{c(u'), c(v_1')\} = \{5, 6\}$ and, since $c(u') \neq c(t_1)$ and $c(v_1') \neq c(t_1)$, $c(t_1) \notin \{5, 6\}$. If $c(t_1') = 4$, then $\{c(u'), c(v_1')\} = \{2, 6\}$ and consequently $c(t_1) \notin \{2, 6\}$. If $c(t_1') = 5$, then $\{c(u'), c(v_1')\} = \{2, 3\}$ and consequently $c(t_1) \notin \{2, 3\}$. Then the only possibilities are $(c(t_1), c(t_1')) = (3, 6)$ or $(c(t_1), c(t_1')) = (4, 6)$. In these cases, $c(u') \neq 6$.

  If $(c(t_1), c(t_1')) = (3, 6)$, then colouring t_2 with 6 and v_2 with a colour in $\{2, 3, 4\} \setminus \{c(u'), c(v_1')\}$, we obtain a (G, T)-colouring, a contradiction. Then $(c(t_1), c(t_1')) =$

$(4,6)$. Consequently, $\{c(u), c(v_1)\} = \{2,6\}$ and, since $\{c(u), c(v_1)\} \cap \{c(u'), c(v_1')\} = \emptyset$, $\{c(u'), c(v_1')\} = \{3,4\}$. This is a contradiction, since $c(t_1) \neq c(u')$ and $c(t_1) \neq c(v_1')$.

- C_e^{int} is isomorphic to A_2 or A_3, $C_{e'}^{int}$ is isomorphic to A_2 or A_3 and $f = v_1v_1'$.

  By minimality of (G,T), there is a $(G - \{u, u', t_2, v_2\} \cup \{rv_1, rv_1'\}, T - \{u, u', t_2, v_2\})$-colouring c which is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$-colouring c such that $c(v_1), c(v_1') \neq 1$. Colour u with some colour in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ and u' with some colour in $Z_6 \setminus (\{1, c(v_1')\} \cup [c(t_1')])$. Then, colour v_2 with 1. If either t_2 is adjacent to at most one in $\{t_1, t_1'\}$ or $\{c(t_1), c(u), c(t_1'), c(u')\} \neq \{3,4,5,6\}$, then we can assign to t_2 a colour in $\{3,4,5,6\}$ not assigned to any of its neighbours to get a (G,T)-colouring, a contradiction.

  So t_2t_1 and t_2t_1' are edges and $\{c(t_1), c(u), c(t_1'), c(u')\} = \{3,4,5,6\}$. By planarity, ru and ru' are not edges and we can recolour u and u' with 1. Then, colouring t_2 with a colour $\alpha \in Z_6 \setminus \{1, 2, c(t_1), c(t_1')\}$ such that $[\alpha] \cup \{1, c(v_1), c(v_1')\} \neq Z_6$ and colouring v_2 with a colour in $Z_6 \setminus ([\alpha] \cup \{1, c(v_1), c(v_1')\})$, we obtain a (G,T)-colouring, a contradiction.

- C_e^{int} is isomorphic to A_2 or A_3, $C_{e'}^{int}$ is isomorphic to A_2 or A_3 and $f = v_2v_2'$.

  By minimality of (G,T), there is a $(G - \{u, u', t_2, v_2\} \cup \{rv_2, rv_2'\}, T - \{u, u', t_2, v_2\})$-colouring which is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$-colouring c such that $c(v_1), c(v_1') \neq 1$. Choose $c(u)$ in $\{2,3\} \setminus \{c(v_2), c(t_2)\}$ and $c(u')$ in $\{2,3\} \setminus \{c(v_2'), c(t_2')\}$ and set $c(v_1) = 1$. If t_1 has at most one neighbour in $\{t_2, t_2'\}$ or $\{c(t_2), c(t_2')\} \neq \{5,6\}$, then we can colour t_1 with a colour in $\{5,6\}$ not appearing on any of its neighbours to get a (G,T)-colouring, a contradiction. Hence t_1 is adjacent to t_2 and t_2' (that is C_e^{int} and $C_{e'}^{int}$) are isomorphic to A_2 and $\{c(t_2), c(t_2')\} = \{5,6\}$. Recolouring u with $c(t_2')$ and u' with $c(t_2)$ and colouring t_1 with 3, we obtain a (G,T)-colouring, a contradiction.

- C_e^{int} is isomorphic to A_2 or A_3, $C_{e'}^{int}$ is isomorphic to A_2 or A_3 and $f = v_1v_2'$.

  By minimality of (G,T), there exists a $G - \{u, u', t_2, v_2\} \cup \{rv_1, rv_2'\}, T - \{u, u', t_2, v_2\})$-colouring c which is a $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$-colouring such that $c(v_1) \neq 1$ and $c(v_2') \neq 1$. Set $c(v_2) = 1$ and colour u with some colour in $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$, u' with some colour in $\{2,3\} \setminus \{c(t_2'), c(v_2')\}$. Note that the set F of forbidden colours for t_2 is the union of $\{1, 2, c(u)\} \cup [c(u')]$ and the set of colours of the neighbours of t_2 in $\{t_1, t_2'\}$. Moreover $F = Z_6$ for otherwise we could colour t_2 with a colour in $Z_6 \setminus F$ to obtain a (G,T)-colouring, a contradiction.

  If $c(u') = 2$, then, since $|F| = 6$, t_2t_1 and t_2t_2' are edges and $\{c(u), c(t_1), c(t_2')\} = \{4,5,6\}$. Since $|c(u) - c(t_1)| \geq 2$, necessarily $\{c(t_1), c(u)\} = \{4,6\}$ and $c(t_2') = 5$. If $c(t_1) = 6$, then recolouring u with a colour in $\{2,3\} \setminus c(v_1)$ and assigning 4 to t_2, we obtain a (G,T)-colouring, a contradiction. Hence $c(t_1) = 4$ and $c(u) = 6$. So $c(v_1) = 2$, and thus $c(v_2') = 3$. Then recolouring u and u' with 1 and v_2 with 4 and setting $c(t_2) = 6$, we obtain a (G,T)-colouring, a contradiction.

  Now, suppose that $c(u') = 3$. Then, since $|F| = 6$, two neighbours of t_2 in $\{u, t_1, t_2'\}$ are coloured 5 and 6. Assume that $c(u) \notin \{5,6\}$, then t_2t_1 and t_2t_2' are edges and $\{c(t_1), c(t_2')\} = \{5,6\}$. Note that, in this case, $c(v_1) \leq 4$ and $c(v_2') \leq 4$. Recolour u and u' with 1, v_2 with 6 and colour t_2 with 3 to get a (G,T)-colouring, a contradiction. Hence $c(u) \in \{5,6\}$. Thus $c(t_1) \leq 4$ and so $c(t_2') \in \{5,6\}$ and $c(v_2') \leq 4$. Thus, t_2t_2' is an edge and $c(t_2') \in (\{5,6\} \setminus \{c(u)\})$. Recolour u' with $c(u)$. If $t_1 \notin N(t_2)$ or $c(t_1) \neq 3$, colouring

t_2 with 3 yields a (G,T)-colouring, a contradiction. So $t_1 \in N(t_2)$ and $c(t_1) = 3$. Then, recolour u and u' with 3 (note that $c(v_1) \geq 5$ and $c(v_2' \neq 3$ as u' was coloured 3) and t_2 with $i \in \{5,6\} \setminus \{c(t_2')\}$. This gives a (G,T)-colouring, a contradiction.

□

Lemmas 15 and 16 immediately imply the following.

Corollary 17 *If f is a penultimate edge, then C_f^{int} is isomorphic to B_1 or B_2, and $rv_1 \in E(G)$. Morever, if $C_f^{int} = B_2$, $rv_3 \notin E(G)$.*

3.3 Antepenultimate edges

We first prove that no antepenultimate edge g has two penultimate predecessors f and f'.

Lemma 18 *Every antepenultimate edge has a unique penultimate predecessor.*

Proof. By contradiction. Suppose that an antepenultimate edge g has two penultimate predecessors f and f'.

According to Corollary 17, C_f^{int} and $C_{f'}^{int}$ are isomorphic to one of the graphs B_1 and B_2. Let us denote the vertices of C_f^{int} by their names in Figure 3 and the vertices of $C_{f'}^{int}$ by their names in Figure 3 augmented with a prime.

Since g, f and f' are bounding the face incident to g in C_g^{int}, the edge g is v_1v_1', v_1v_3', v_3v_3' or v_3v_1'. Since rv_1 and rv_1' are edges, then $g = v_1v_1'$, for otherwise rv_1 would cross g.

First, suppose that $C_{f'}^{int}$ is isomorphic to B_1, i.e., $rv_2' \in E$. By minimality of (G,T), there is a $((G-\{v_2,v_3\})\cup\{v_2'u, v_2'v_1\}, T-\{v_2,v_3\})$-colouring, which is a $(G-\{v_2,v_3\}, T-\{v_2,v_3\})$-colouring such that $c(v_2') \notin \{c(u), c(v_1)\}$. Setting $c(v_2) = c(v_2')$ and $c(v_3) = 1$ gives a (G,T)-colouring, a contradiction.

The case C_f^{int} is isomorphic to B_1 is symmetric, so we may assume that both are C_e^{int} and $C_{e'}^{int}$ are isomorphic to B_2. By minimality of (G,T), there exists a $(G - \{v_2, v_2', v_3, t_2\}, T - \{v_2, v_2', v_3, t_2\})$-colouring. Set $c(v_2) = c(v_2') = 1$. Then, one can choose $c(t_2)$ in $L = Z_6 \setminus \{1, 2, c(u), c(u')\}$ such that $I = [c(t_2)] \cup \{1, c(v_1), c(v_1')\} \neq Z_6$ because $|L| \geq 2$. Hence colouring v_3 with a colour in $Z_6 \setminus I$, we obtain a (G,T)-colouring, a contradiction. □

From Lemma 18, for every antepenultimate edge g, g has only one predecessor f (which must be penultimate), or g has two predecessors: a penultimate edge f and an ultimate edge e'. From Lemmas 15 and 14, C_f^{int} is B_1 or B_2, and $C_{e'}^{int}$ is A_1, A_2 or A_3.

To deal with theses cases, we need the following two auxiliary lemmas.

Lemma 19 *Suppose that (G,T) contains a configuration isomorphic to B_1 (see Figure 3). If there is a $(G - \{u, v_2\}, T - \{u, v_2\})$-colouring c satisfying one of the following conditions :*

(a) $c(t_2) = 6$ and $(c(t_1), c(v_3)) \neq (5, 4)$;

(b) $c(v_3) = 1$ and $c(t_1) \neq c(t_2)$;

Then there is a (G,T)-colouring.

Proof. Let $L(u) = Z_6\backslash([c(t_1)]\cup\{1, c(v_1)\})$ and $L(v_2) = Z_6\backslash([c(t_2)]\cup\{1, c(v_1), c(v_3)\})$ be the set of colours available for u and v_2 respectively. Clearly $L(u) \neq \emptyset$. Observe that the conditions (a) and (b) also imply that $L(v_2) \neq \emptyset$. So, if $|L(u)| \geq 2$, $|L(v_2)| \geq 2$ or $L(u) \neq L(v_2)$, one can choose distinct colours $c(u) \in L(u)$ and $c(v_2) \in L(v_2)$ to obtain a (G, T)-colouring. It is a simple matter to check that in both cases these conditions are satisfied. $\square$

Lemma 20 *Suppose that (G, T) contains a configuration isomorphic to B_2 (see Figure 3). If there is a $(G - \{u, v_2\}, T - \{u, v_2\})$-colouring c satisfying one of the following conditions :*

(a) $c(t_1) = c(t_2)$ *and* $c(v_3) \neq 1$*;*

(b) $c(t_1) \neq c(t_2)$ *and*

- *(b1)* $c(t_1) = 6$*; or*
- *(b2)* $c(v_1) = c(t_2)$*; or*
- *(b3)* $c(t_2) \in [c(t_1)]$.

Then G has a (G, T)-colouring.

Proof. Let $L(u) = Z_6 \setminus \{1, [c(t_1)], c(t_2), c(v_1)\}$ and $L(v_2) = Z_6 \setminus \{[c(t_2)], c(v_1), c(v_3)\}$ be the set of colours available for u and v_2 respectively. Clearly $L(v_2) \neq \emptyset$. Observe that the conditions (a), (b1), (b2) and (b3) also imply that $L(u) \neq \emptyset$. So, if $|L(u)| \geq 2$, $|L(v_2)| \geq 2$ or $L(u) \neq L(v_2)$, one can choose distinct colours $c(u) \in L(u)$ and $c(v_2) \in L(v_2)$ to obtain a (G, T)-colouring. It is a simple matter to check that in each case these conditions are satisfied. $\square$

Now we prove that the case of an antepenultimate edge g with a penultimate predecessor f and an ultimate predecessor e' is impossible.

Lemma 21 *Every antepenultimate edge has a unique predecessor.*

Proof. By contradiction. Suppose that an antepenultimate edge g has two predecessors f and f'. By Lemma 18, one of those is not penultimate. So, wihtout loss of generality, f is penultimate, and f' is not. Hence f' is minimal.

According to Corollary 17, C_f^{int} is isomorphic to B_1 or B_2, and according to Lemma 14, $C_{f'}^{int}$ is isomorphic to some of A_1, A_2 or A_3. Let us denote the vertices of C_f^{int} by their names in Figure 3 and the vertices of $C_{f'}^{int}$ by their names in Figure 2 augmented with a prime.

Since g, f and f' are bounding the face incindent to g in C_g^{int}, the edge g is v_1v_1', v_1v_2', v_3v_1' or v_3v_2'. Moreover, since rv_1 is an edge, rv_3 is not an edge if C_f^{int} is isomorphic to B_2 , and rv_1' is an edge if $C_{f'}^{int}$ is isomorphic to A_1, we must be in one of the following cases:

- C_f^{int} is isomorphic to B_1, $C_{f'}^{int}$ is isomorphic to A_2 or A_3 and $g = v_1v_2'$.

  By minimality of (G, T), there is a $(G - \{v_2, t_2, u', v_3\} \cup \{v_2'r\}, T - \{v_2, t_2, u', v_3\})$-colouring c which is a $(G - \{v_2, t_2, u', v_3\}, T - \{v_2, t_2, u', v_3\})$-colouring such that $c(v_2') \neq 1$. Set $c(v_3) = 1$. Let $L(t_2) \supseteq Z_6 \setminus \{1, 2, c(t_2')\}$, $L(v_2) = Z_6 \setminus \{1, c(u), c(v_1)\}$ and $L(u') = Z_6\setminus\{1, c(t_2'), c(v_2')\}$. Clearly, there exists at most one $i \in Z_6$ such that $L(u') = [i]$ and at most one $j \in Z_6$ such that $L(v_2) = [j]$. Thus, as $|L(t_2)| \geq 3$, there exists $k \in L(t_2)$

such that $L(u') \setminus [k] \neq \emptyset$ and $L(v_2) \setminus [k] \neq \emptyset$. Setting $c(t_2) = k$ and colouring u' and v_2 by colours in $L(u') \setminus [k]$ and $L(v_2) \setminus [k]$, respectively, we obtain a (G, T)-colouring, a contradiction.

- C_f^{int} is isomorphic to B_2, $C_{f'}^{int}$ is isomorphic to A_2 or A_3 and $g = v_1v_2'$.

  By minimality of (G, T), there is a $((G - \{v_2, t_2, v_3, u'\}) \cup \{t_2'u, t_2'v_1\}, T - \{v_2, t_2, v_3, u'\})$-colouring c which is a $(G - \{v_2, t_2, v_3, u'\}, T - \{v_2, t_2, v_3, u'\})$-colouring such that $c(t_2') \notin \{c(u), c(v_1)\}$.

  Suppose that $t_2t_2' \in E(G)$. If we can colour t_2 with $\beta \in [c(v_1)] \cup \{6\}$, then we can colour u' with some colour in $Z_6 \setminus (\{c(t_2'), c(v_2')\} \cup [\beta])$, v_3 with some colour in $Z_6 \setminus (\{c(u'), c(v_2'), c(v_1)\} \cup [\beta])$ and v_2 with some colour in $Z_6 \setminus (\{c(v_3), c(u), c(v_1)\} \cup [\beta])$, a contradiction. So, there is no available colour in $[c(v_1)] \cup \{6\}$ for t_2; that is, $[c(v_1)] \cup \{6\} \subseteq \{1, 2, c(u), c(t_2')\}$. Since $c(v_1) \notin \{1, c(u), c(t_2')\}$, we must have $c(v_1) = 2$ and $\{c(u), c(t_2')\} = \{3, 6\}$. colour u' with 2 (since $v_2' \in N(v_1)$ we know that $c(v_2') \neq c(v_1)$), v_2 with 1 and v_3 with $c(t_2')$. Colour t_2 with 4 if $c(u) = 3$ and with 5 otherwise. This gives a (G, T)-colouring, a contradiction.

  Now, suppose that $ru' \in E(G)$. If $c(u) \neq 6$, then we can colour t_2 with 6 and u', v_3 and v_2 can be greedily coloured in this order, a contradiction; thus, $c(u) = 6$. Let $L(u') = Z_6 \setminus \{1, c(t_2'), c(v_2')\}$ be the colours available for u'; note that if $L(u') = [i]$ for some $i \in Z_6$ then $c(t_2') = 6$ and $c(v_2') = 2$, a contradiction since $c(t_2') \neq c(u)$. Clearly, there exists $\beta \in [c(v_1)] \setminus \{1, 2\}$ so we can colour t_2 with β, u' with any colour in $L(u') \setminus [\beta]$ (recall that $L(u') \neq [i]$ for all $i \in Z_6$). Then colour v_3 and v_2 greedily gives a (G, T)-colouring, a contradiction.

- C_f^{int} is isomorphic to B_1 or B_2, $C_{f'}^{int}$ is isomorphic to A_2 or A_3 and $g = v_1v_1'$.

  By minimality of (G, T), there is a $((G - \{t_2, v_2, v_3\}) \cup \{uv_1', ru', rv_1'\}, T - \{t_2, v_2, v_3\}$-colouring, which is a $(G - \{t_2, v_2, v_3\}, T - \{t_2, v_2, v_3\})$-colouring such that $c(v_1'), c(u') \neq 1$ and $c(v_1') \neq c(u)$.

  Suppose that we can colour t_2 with $\beta \in [c(v_1)] \cup \{6\}$. We know that there is at least one colour $i \in Z_6 \setminus (\{1, c(u), c(v_1)\} \cup [\beta])$ available for v_2 and at least one colour $j \in Z_6 \setminus (\{c(v_1'), c(u'), c(v_1)\} \cup [\beta])$ available for v_3. Since $c(v_1') \notin \{1, c(u)\}$, then $i \neq j$ and we can colour v_2 with i and v_3 with j to obtain a (G, T)-colouring, a contradiction.

  So, suppose that the colours of $[c(v_1)] \cup \{6\}$ all appear in $N(t_2)\{v_2, v_3\}$; since $|([c(v_1)] \cup \{6\}) \setminus \{1, 2\}| \geq 2$, t_2 must be adjacent to at least one of u and t_1'.

  Assume first $t_2t_1' \in E$. Then recolour u' with 1. If $ut_2 \notin E$ or $[c(v_1)] \cup \{6\} \not\subseteq \{1, 2, c(u), c(t_1')\}$, note that we can apply the same argument as before since it holds even if $c(u') = 1$; so suppose otherwise. In this case we must have: either (a) $c(v_1) = 6$, $c(u) = 5$ and $c(t_1') = 6$; or (b) $c(v_1) = 2$ and $\{c(u), c(t_1') = \{3, 6\}$. colour v_2 with 1. If (a) occurs, then colour t_2 with 3 and v_3 with 5; if (b) occurs and $c(t_1') = 6$, then colour t_2 with 4 and v_3 6; if (b) occurs and $c(t_1') = 3$, then colour t_2 with 5 and v_3 with 3.

  Hence $t_2t_1' \notin E$, and so $t_2u \in E$. The possible situations are: (c) $c(v_1) = 6$, $c(u) = 5$ and $c(u') = 6$; or (d) $c(v_1) = 2$ and $\{c(u), c(u')\} = \{3, 6\}$. If (c) occurs, then colour v_3 with $\{2, 5\} \setminus \{c(v_1')\}$ and t_2 with $\{3, 4\} \setminus [c(v_3)]$. If (d) occurs, then colour v_3 with $c(u)$ (recall that $c(v_1') \neq c(u)$) and t_2 with $\{4, 5\} \setminus [c(v_3)]$. In both cases we get a (G, T)-colouring, a contradiction.

- C_f^{int} is isomorphic to B_1, $C_{f'}^{int}$ is isomorphic to A_1 and $g = v_1v_1'$.

  Let (G', T') be the graph pair obtained from $(G - \{u, v_2, t_2, v_3, u'\}, T - \{u, v_2, t_2, v_3, u'\})$ by identifying t_1 and t_1'. By minimality of (G, T), there exists a (G', T')-colouring which is a $(G - \{u, v_2, t_2, v_3, u'\}, T - \{u, v_2, t_2, v_3, u'\})$ -colouring such that $c(t_1) = c(t_1')$. Set $c(t_2) = 6$ and $c(u') = c(v_1)$. Let $L = \{2, 3, 4\} \setminus \{c(v_1), c(v_1')\}$.

  If $c(t_1) \neq 5$, then choosing $c(v_3) \in L$, and applying Lemma 19, we obtain a (G, T)-colouring, a contradiction. Hence $c(t_1) = c(t_1') = 5$.

  If $L \neq \{4\}$, then we can choose $c(v_3) \in L \setminus \{4\}$, and apply Lemma 19 to get a (G, T)-colouring, a contradiction. Hence $\{c(v_1), c(v_1')\} = \{2, 3\}$.

  Now setting $c(v_3) = 5$, $c(v_2) = 6$, $c(u) = c(v_1')$ and recolouring t_2 with 3, we obtain a (G, T)-colouring, a contradiction.

- C_f^{int} is isomorphic to B_1, $C_{f'}^{int}$ is isomorphic to A_1 and $g = v_3v_1'$.

  Let (G', T') be the graph pair obtained from $(G - \{v_1, v_2, u, t_1, u'\}, T - \{v_1, v_2, u, t_1, u'\})$ by identifying t_2 and t_1'. By minimality of (G, T), there exists a (G', T')-colouring which is a $(G - \{v_1, v_2, u, t_1, u'\}, T - \{v_1, v_2, u, t_1, u'\})$-colouring such that $c(t_2) = c(t_1')$. Set $c(t_1) = 6$.

  If $c(t_2) = c(t') = 6$, then set $c(u) = c(v_3)$. One can then greedily extend the colouring to v_1, v_2 and u' in this order, a contradiction.

  If $c(t_2) \neq 6$, then one can choose $c(v_1) \in [c(t_2)] \setminus \{5, 6\}$. This is valid since $c(v_3)$ and $c(v_1')$ are not in $[c(t_2)]$. On can then greedily extend the colouring to v_2, u and u' in this order, a contradiction.

- C_f^{int} is isomorphic to B_2, $C_{f'}^{int}$ is isomorphic to A_1 and $g = v_3v_1'$.

  By minimality of (G, T), there exists a $(G - \{t_1, u, v_1, v_2\} \cup \{rv_3\}, T - \{t_1, u, v_1, v_2\})$-colouring c. Set $c(v_2) = 1$ and let $L(v_1) = Z_6 \setminus \{1, c(v_3), c(u'), c(v_1')\}$ be the colours available for v_1.

  If $L(v_1) \neq \{5, 6\}$, then colouring t_1 with 6, v_1 with some colour in $L(v_1) \setminus \{5, 6\}$ and u with some colour in $Z_6 \setminus \{1, 5, 6, c(v_1), c(t_2)\}$, we obtain a (G, T)-colouring, a contradiction.

  If $L(v_1) = \{5, 6\}$, then $c(u'), c(v_1') \in \{2, 3, 4\}$ and consequently $c(t_1') \in \{5, 6\}$. We can suppose that $c(t_1') = 5$ and $c(v_3) = 4$ for otherwise we can recolour u' with $c(v_3)$ and fall in the case $L(v_1) \neq \{5, 6\}$. So, $\{c(u'), c(v_1')\} = \{2, 3\}$ and $c(t_2) \geq 6$. Setting $c(t_1) = 3$, $c(u) = 5$ and $c(v_1) = 6$, we obtain a (G, T)-colouring, a contradiction.

□

The next two lemmas prove that the case of an antepenultimate edge g with only one predecessor f, which must be penultimate, is also impossible. Lemma 22 prove for $C_f^{int} = B_1$ and Lemma 23 prove for $C_f^{int} = B_2$.

Lemma 22 *There is no antepenultimate edge g with only one penultimate predecessor f such that C_f^{int} is B_1.*

Proof. One of the endvertices of g must be v_1 or v_3 (see Figure 3). We now distinguish some cases depending on the possible endvertices of g.

(a) Assume $g = v'v_3$ with v' a leaf with twig t'. Since $rv_1 \in E(G)$, by planarity, $t' \neq t_1$. Let (G', T') be the graph pair obtained from $(G - \{t_1, v_1, u, v_2\}, T - \{t_1, v_1, u, v_2\})$ by identifying t' and t_2. By minimality of (G, T), there is a (G', T')-colouring which is a $(G - \{t_1, v_1, u, v_2\}, T - \{t_1, v_1, u, v_2\})$-colouring such that $c(t') = c(t_2)$. Set $c(v_2) = c(v')$ and $c(t_1) = 6$.

If $c(t') \in \{5, 6\}$, then setting $c(u) = c(v_3)$ and choosing $c(v_1)$ in $\{2, 3, 4\} \setminus \{c(v'), c(v_3)\}$, we obtain a (G, T)-colouring, a contradiction.

If $c(t') \in \{3, 4\}$, then setting $c(v_1) = c(t') - 1$ and choosing $c(u)$ in $\{2, 3, 4\} \setminus \{c(v_1), c(v_2)\}$, we obtain a (G, T)-colouring, a contradiction.

(b) Assume $g = t'v_3$ with t' a twig. We can apply an argument similar to (a) choosing $c(v_2) \in Z_6 \setminus (\{1, c(v_3)\} \cup [c(t_2)])$.

(c) Assume $g = v_1 r$. Since G is triangulated, the edge $v_1 t_2$ must exist. This is a contradiction, since f is the sucessor of e.

(d) Assume $g = v_1 t_2$. Then v_3 is a leaf of degree at most 3, a contradiction from Lemma 7.

(e) Assume $g = v_1 v'$ with v' a leaf with twig t_2. Let (G', T') be the graph pair obtained from $(G - \{v_3\}, T - \{v_3\})$ by identifying v_2 and v'. By minimality of (G, T), there is a (G', T')-colouring which is a $(G - \{v_3\}, T - \{v_3\})$-colouring such that $c(v_2) = c(v')$. Hence one can colour v_3 with a colour from $Z_6 \setminus \{[c(t_2)], c(v_2), c(v_1)\}$ to obtain a (G, T)-colouring, a contradiction.

(f) Assume $g = v_1 t'$ with $t' \neq t_2$ a twig. Since g is the succesor of f, $v_1 t_2$ is not an edge and $v_1 r$ is not inside C_g, so $v_3 t' \in E$.

Assume first that $rv_3 \notin E(G)$. By minimality of (G, T), there is a $((G - \{u, v_2, v_3\}) \cup t_1 t_2, T - \{u, v_2, v_3\})$-colouring which is a $(G - \{u, v_2, v_3\}, T - \{u, v_2, v_3\})$-colouring such that $c(t_1) \neq c(t_2)$. Since $c(t'), c(v_1) \neq 1$, we can colour v_3 with 1. Then, by Lemma 19 (b), there is a (G, T)-colouring, a contradiction.

Assume now that $rv_3 \in E(G)$. Then $t't_2 \notin E$ by planarity. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ by identifying t_1 and t'. By minimality of (G, T), there is a (G', T')-colouring, which is a $((G - \{u, t_2, v_2, v_3\}), T - \{u, t_2, v_2, v_3\})$-colouring such that $c(t_1) = c(t')$. Set $c(t_2) = 6$. One can choose $c(v_3) \in \{2, 3\} \setminus \{c(t'), c(v_1)\}$ because $|c(v_1) - c(t')| \geq 2$. Then by Lemma 19 (a), there is a (G, T)-colouring, a contradiction.

(g) Assume $g = v_1 v'$ with v' a leaf with twig $t' \neq t_2$. Since g is the succesor of f, $v_1 t_2$ is not an edge and $v_1 r$ and $v_1 t'$ are not inside C_g, so $v_3 t' \in E$.

Assume first that $rv_3 \notin E(G)$. By minimality of (G, T), there is a $((G - \{u, t_2, v_2, v_3\}) \cup rv', T - \{u, t_2, v_2, v_3\})$ -colouring which is a $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$-colouring such that $c(v') \neq 1$. Since $c(t'), c(v'), c(v_1) \neq 1$, we can colour v_3 with 1. Then, colouring t_2 with a colour from $Z_6 \{1, 2, c(t'), c(v'), c(t_1)\}$ and using Lemma 19 (b), we obtain a (G, T)-colouring, a contradiction.

Assume now that $rv_3 \notin E(G)$. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ by identifying t_1 and t'. By minimality of (G, T), there is a (G', T')-colouring, which is a $((G - \{u, t_2, v_2, v_3\}), T - \{u, t_2, v_2, v_3\})$-colouring such that $c(t_1) = c(t')$. If $\{c(v_1), c(v')\} \neq \{2,3\}$, then one can choose $c(v_3)$ in $\{2,3\} \setminus \{c(t'), c(v'), c(v_1)\}$. Then setting $c(t_2) = 6$ and applying Lemma 19 (a), we obtain a (G,T)-colouring, a contradiction. Thus $\{c(v_1), c(v')\} = \{2,3\}$, and so $c(t_1) \geq 5$. Setting $c(u) = c(v')$, $c(t_2) = 3$ and choosing $c(v_3)$ in $\{5,6\} \setminus c(t')$ and $c(v_2)$ in $\{5,6\} \setminus c(v_3)$ yields a (G,T)-colouring, a contradiction.

□

Lemma 23 *There is no antepenultimate edge g with only one penultimate predecessor f such that C_f^{int} is B_2.*

Proof. One of the endvertices of g must be v_1 or v_3 (see Figure 3). We now distinguish some cases depending on the possible endvertices of g.

(a) Assume $g = v'v_3$ with v' a leaf with twig t'. Since $rv_1 \in E(G)$, by planarity, $t' \neq t_1$. By minimality of (G,T), there is a $((G - \{t_1, v_1, u\}) \cup \{t_2v', t_2t'\}, T - \{t_1, v_1, u\})(G', T')$-colouring which is a $(G - \{t_1, v_1, u\}, T - \{t_1, v_1, u\})$-colouring such that $c(v') \neq c(t_2)$ and $c(t') \neq c(t_2)$. Hence one can colour v_1 with $c(t_2)$ and colour t_1 with a colour in $Z_6 \setminus ([c(t_2)] \cup \{1,2\})$. From Lemma 20 (b2), we obtain a (G,T)-colouring, a contradiction.

(b) Assume $g = t'v_3$ with t' a twig. We can apply an argument similar to (a).

(c) Assume $g = v_1r$. Since G is triangulated, the edge v_1t_2 must exist. This is a contradiction, since f is the sucessor of e.

(d) Assume $g = v_1t_2$. Then v_3 is a leaf of degree at most 3, a contradiction from Lemma 7.

(e) Assume $g = v_1v'$ with v' a leaf adjacent to t_2 in T. Since f is the successor of v_1v_3, $v_1t_2 \notin E(G)$ and so $v_2v_3 \in E(G)$ because G is triangulated. Let (G', T') be the graph pair obtained from $(G - \{v_3\}, T - \{v_3\})$ by identifying v_2 and v'. By minimality of (G,T), there is a (G', T')-colouring which is a $(G - \{v_3\}, T - \{v_3\})$-colouring such that $c(v_2) = c(v')$. Hence one can colour v_3 with a colour from $Z_6 \setminus \{[c(t_2)], c(v_2), c(v_1)\}$ to obtain a (G,T)-colouring, a contradiction.

(f) Assume $g = v_1t'$ with $t' \neq t_2$ a twig. Let (G', T') be the graph pair obtained from $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ by identifying t_1 and t'. This is possible since t_1t' is not an edge by planarity. By minimality of (G,T), there is a (G', T')-colouring which is a $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$-colouring such that $c(t') = c(t_1)$. Let $L(t_2) = Z_6 \setminus \{1, 2, c(t')\}$. If $c(t_1) = 6$, then, colouring v_3 with 1 and t_2 with a colour from $L(t_2) \setminus \{6\}$, and using Lemma 20 (b1), we obtain a (G,T)-colouring, a contradiction. So $c(t_1) \neq 6$, that is $c(t_1) \in \{3,4,5\}$. We can colour t_2 with a colour from $[c(t_1)] \setminus \{c(t'), c(t_1)\} \subseteq L(v_2)$. By Lemma 20 (b3), there is a (G,T)-colouring, a contradiction.

(g) Assume $g = v_1v'$ with v' a leaf with twig $t' \neq t_2$. By minimality of (G,T), there is a $((G - \{u, t_2, v_2, v_3\}) \cup \{v_1t'\}, T - \{u, t_2, v_2, v_3\})(G', T')$-colouring which is a $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$-colouring such that $c(v_1) \neq c(t')$. If $c(v_1) \neq 2$, we can colour t_2 with $c(v_1)$, since $c(t'), c(v') \neq c(v_1)$. Then, colouring v_3 with a colour from $Z_6 \setminus \{[c(v_1)], c(t'), c(v')\}$, and applying Lemma 20 (b2), we obtain a (G,T)-colouring, a contradiction. So, $c(v_1) = 2$.

Suppose that $c(v') \neq 1$. Since $c(t'), c(v') \notin \{1,2\}$, then $\{c(t'), c(v')\} \in \{\{3,5\},\{3,6\},\{4,6\}\}$. Let $L(v_3) = Z_6 \setminus \{2, c(t'), c(v')\}$ and let $L(t_2) = Z_6 \setminus \{1, 2, c(t'), c(v')\}$. If $L(v_2) \cap ([c(t_1)] \setminus \{c(t_1)\}) \neq \emptyset$, then chossing $c(t_2)$ in $[c(t1)] \setminus \{c(t_1)\}$ and $c(v_3)$ in $L(v_3) \setminus [c(t_2)]$ (observe that $|[c(t_2)] \cap L(v_3)| \leq 2$, since $L(v_3)$ has no three consecutive integers), and using Lemma 20 (b3), we obtain a (G,T)-colouring, a contradiction. Then $L(v_2) \cap ([c(t_1)] \setminus \{c(t_1)\}) = \emptyset$. If $c(t_1) = 3$, then $\{c(t'), c(v')\} = \{4,6\}$. In this case, colouring t_2 with 3, v_3 and u with 5 and v_2 with 6, we can obtain a (G,T)-colouring, a contradiction. If $c(t_1) = 4$, then $\{c(t'), c(v')\} = \{3,5\}$, and if $c(t_1) = 5$, then $\{c(t'), c(v')\} = \{4,6\}$. In both cases, setting $c(t_2) = c(t_1)$, choosing $c(v_3)$ in $Z_6 \setminus \{1, 2, [c(t_1)]\}$ and using Lemma 20 (a), we obtain a (G,T)-colouring, a contradiction.

Hence $c(v') = 1$. If $c(t_1) \in \{3,4\}$, colour t_2 with 3 (if $c(t') \neq 3$) or 4 (otherwise). If $c(t_1) = 5$, colour t_2 with 6 (if $c(t') \neq 6$) or 5 (otherwise). These cases satisfy the conditions $c(t_2) \in [c(t_1]$ and $Z_6 \setminus \{1, 2, c(t'), [c(t_2)]\} \neq \emptyset$. Then, colouring v_3 with a colour from $Z_6 \setminus \{1, 2, c(t'), [c(t_2)]\}$, and using Lemma 20 ((a) or (b3)), we obtain (G,T)-colouring, a contradiction.

□

Lemmas 18, 21, 22 and 23 directly imply the following.

Corollary 24 (G,T) *has no antepenultimate edges.*

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